

JEE (ADVANCED)—2017

MATHEMATICS PAPER-II

SECTION I

- This section contains **Seven** questions.
- Each question has **Four** options (a), (b), (c) and (d). **Only One** of these options is correct.
- For each question, marks will be awarded in one of the following categories:

<i>Full marks</i>	:	+ 3	If only the bubble corresponding to the correct option is darkened.
<i>Zero marks</i>	:	0	If none of the bubbles is darkened
<i>Negative marks</i>	:	-1	In all other cases

- | | |
|--|---|
| <p>1. How many 3×3 matrices M with entries from $\{0, 1, 2\}$ are there, for which the sum of the diagonal entries of $M^T M$ is 5?
 (a) 135 (b) 198
 (c) 162 (d) 126</p> <p>2. Three randomly chosen nonnegative integers x, y and z are found to satisfy the equation $x + y + z = 10$. Then the probability that z is even, is
 (a) $\frac{5}{11}$ (b) $\frac{6}{11}$
 (c) $\frac{1}{2}$ (d) $\frac{36}{55}$</p> <p>3. The equation of the plane passing through the point $(1, 1, 1)$ and perpendicular to the planes $2x + y - 2z = 5$ and $3x - 6y - 2z = 7$, is
 (a) $14x + 2y - 15z = 1$
 (b) $14x - 2y + 15z = 27$
 (c) $-14x + 2y + 15z = 3$
 (d) $14x + 2y + 15z = 31$</p> <p>4. If $y = y(x)$ satisfies the differential equation
 $8\sqrt{x}(\sqrt{9+\sqrt{x}})dy = (\sqrt{4+\sqrt{9+\sqrt{x}}})^{-1} dx, x > 0$
 and $y(0) = \sqrt{7}$, then $y(256) =$
 (a) 3 (b) 16
 (c) 9 (d) 80</p> | <p>5. If $f: \mathbf{R} \rightarrow \mathbf{R}$ is a twice differentiable function such that $f''(x) > 0$ for all $x \in \mathbf{R}$, and $f\left(\frac{1}{2}\right) = \frac{1}{2}$, $f(1) = 1$, then
 (a) $f'(1) \leq 0$
 (b) $\frac{1}{2} < f'(1) \leq 1$
 (c) $f'(1) > 1$
 (d) $0 < f'(1) \leq \frac{1}{2}$</p> <p>6. Let $S = \{1, 2, 3, \dots, 9\}$. For $k = 1, 2, \dots, 5$, let N_k be the number of subsets of S, each containing five elements out of which exactly k are odd. Then $N_1 + N_2 + N_3 + N_4 + N_5 =$
 (a) 126 (b) 252
 (c) 210 (d) 125</p> <p>7. Let O be the origin and let PQR be an arbitrary triangle. The point S is such that
 $\mathbf{OP} \cdot \mathbf{OQ} + \mathbf{OR} \cdot \mathbf{OS} = \mathbf{OR} \cdot \mathbf{OP} + \mathbf{OQ} \cdot \mathbf{OS}$ $= \mathbf{OQ} \cdot \mathbf{OR} + \mathbf{OP} \cdot \mathbf{OS}$ Then triangle PQR has S as its
 (a) circumcentre (b) incentre
 (c) centroid (d) orthocenter</p> |
|--|---|

SECTION - 2

- This section contains **Seven** questions.
- Each question has **Four** options (a), (b), (c) and (d). **One or More than One** of these four options is(are) correct.
- For each question, marks will be awarded in one of the following categories:

<i>Full marks</i>	:	+ 4	If only the bubble(s) corresponding to all the correct option (s) is (are) darkened.
<i>Partial marks</i>	:	+ 1	For darkening a bubble corresponding to each correct option , provided NO incorrect option is darkened

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Zero marks : 0 If none of the bubbles is darkened
 Negative marks : -2 In all other cases

- For example, if (a), (c) and (d) are all the correct options for a question, darkening all these three will get +4 marks; darkening only (a) and (d) will get +2 marks; and darkening (a) and (b) will get -2 marks, as a wrong option is also darkened.

8. If $I = \sum_{k=1}^{98} \int_k^{k+1} \frac{k+1}{x(x+1)} dx$, then

- (a) $I < \frac{49}{50}$ (b) $I > \frac{49}{50}$
 (c) $I < \log_e 99$ (d) $I > \log_e 99$

9. If the line $x = \alpha$ divides the area of region $R = \{(x, y) \in \mathbf{R}^2 : x^3 \leq y \leq x, 0 \leq x \leq 1\}$ into two equal parts, then

- (a) $\alpha^4 + 4\alpha^2 - 1 = 0$
 (b) $0 < \alpha \leq \frac{1}{2}$
 (c) $2\alpha^4 - 4\alpha^2 + 1 = 0$
 (d) $\frac{1}{2} < \alpha < 1$

10. If $g(x) = \int_{\sin x}^{\sin(2x)} \sin^{-1}(t) dt$, then

- (a) $g'\left(\frac{\pi}{2}\right) = 2\pi$
 (b) $g'\left(-\frac{\pi}{2}\right) = 2\pi$
 (c) $g'\left(\frac{\pi}{2}\right) = -2\pi$
 (d) $g'\left(-\frac{\pi}{2}\right) = -2\pi$

11. If $f(x) = \begin{vmatrix} \cos(2x) & \cos(2x) & \sin(2x) \\ -\cos x & \cos x & -\sin x \\ \sin x & \sin x & \cos x \end{vmatrix}$, then

- (a) $f(x)$ attains its maximum at $x = 0$
 (b) $f(x)$ attains its minimum at $x = 0$

(c) $f'(x) = 0$ at exactly three points in $(-\pi, \pi)$

(d) $f'(x) = 0$ at more than three points in $(-\pi, \pi)$

12. Let α and β be nonzero real numbers such that $2(\cos \beta - \cos \alpha) + \cos \alpha \cos \beta = 1$. Then which of the following is/are true?

(a) $\tan\left(\frac{\alpha}{2}\right) + \sqrt{3} \tan\left(\frac{\beta}{2}\right) = 0$

(b) $\tan\left(\frac{\alpha}{2}\right) - \sqrt{3} \tan\left(\frac{\beta}{2}\right) = 0$

(c) $\sqrt{3} \tan\left(\frac{\alpha}{2}\right) - \tan\left(\frac{\beta}{2}\right) = 0$

(d) $\sqrt{3} \tan\left(\frac{\alpha}{2}\right) + \tan\left(\frac{\beta}{2}\right) = 0$

13. Let $f(x) = \frac{1-x(1-|1-x|)}{|1-x|} \cos\left(\frac{1}{1-x}\right)$, for $x \neq 1$.

Then

(a) $\lim_{x \rightarrow 1^-} f(x)$ does not exist

(b) $\lim_{x \rightarrow 1^-} f(x) = 0$

(c) $\lim_{x \rightarrow 1^+} f(x) = 0$

(d) $\lim_{x \rightarrow 1^+} f(x)$ does not exist

14. If $f: \mathbf{R} \rightarrow \mathbf{R}$ is a differentiable function such that $f'(x) > 2f(x)$ for all $x \in \mathbf{R}$, and $f(0) = 1$, then

(a) $f'(x) < e^{2x}$ in $(0, \infty)$

(b) $f(x)$ is increasing in $(0, \infty)$

(c) $f(x)$ is decreasing in $(0, \infty)$

(d) $f(x) > e^{2x}$ in $(0, \infty)$

SECTION - 3

- This section contains **Two** paragraphs.
- Based on each paragraph, there are **Two** questions.
- Each questions has **Four** options (a), (b), (c) and (d). **Only One** of these four options is correct.
- For each question, marks will be awarded in one of the following categories:

Full marks : +3 If only the bubble corresponding to the correct option is darkened.

Zero marks : 0 In all other cases

Paragraph-1

Let O be the origin, and $\mathbf{OX}, \mathbf{OY}, \mathbf{OZ}$ be three unit vectors in the directions of the sides $\mathbf{QR}, \mathbf{RP}, \mathbf{PQ}$ respectively, of a triangle PQR .

15. If the triangle PQR varies, then the minimum value of $\cos(P + Q) + \cos(Q + R) + \cos(R + P)$ is

- (a) $\frac{3}{2}$ (b) $\frac{5}{3}$
- (c) $-\frac{5}{3}$ (d) $-\frac{3}{2}$

16. $|\mathbf{OX} \times \mathbf{OY}| =$

- (a) $\sin(P + R)$ (b) $\sin(Q + R)$
- (c) $\sin(P + Q)$ (d) $\sin 2R$

Paragraph-2

Let p, q be integers and let α, β be the roots of the equation, $x^2 - x - 1 = 0$, where $\alpha \neq \beta$. For $n = 0, 1, 2, \dots$, let $\alpha_n = p\alpha^n + q\beta^n$.

FACT : If a and b are rational numbers and $a + b\sqrt{5} = 0$, then $a = 0 = b$.

17. If $a_4 = 28$, then $p + 2q =$

- (a) 12 (b) 14
- (c) 7 (d) 21

18. $a_{12} =$

- (a) $a_{11} + 2a_{10}$ (b) $a_{11} - a_{10}$
- (c) $2a_{11} + a_{10}$ (d) $a_{11} + a_{10}$

Answers

Section-1

- 1. (b) 2. (b) 3. (d)
- 4. (a) 5. (c) 6. (a)
- 7. (d)

Section-2

- 8. (b), (c) 9. (c), (d)
- 10. None of the given answer is correct
- 11. (a), (c) 12. (a), (b) 13. (b), (d)
- 14. (b), (d)

Section-3

- 15. (d) 16. (c) 17. (a)
- 18. (d)

Hints and Solutions

1. Let $M = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$

$$M^T M = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

Sum of the diagonal elements of $M^T M$ is $d = a_1^2 + a_2^2 + a_3^2 + b_1^2 + b_2^2 + b_3^2 + c_1^2 + c_2^2 + c_3^2$

where $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3 \in \{0, 1, 2\}$

Now, $d = 5$, if exactly 5 of

$a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3$ are 1's and rest 4 of them are zeros

This 5 numbers can be chosen in ${}^9C_5 = 126$ ways

Also, $d = 5$, if exactly one of

$a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3$ is 2 and exactly one of it is 1.

We can choose two numbers in 9C_2 and arrange them in two ways, that is, in $({}^9C_2)(2) = 72$ ways

Hence, the number of ways is 198.

2. Number of non-negative integral solutions of $x + y + z = 10$ is ${}^{12}C_2 = 66$.

We now count the number of ways in which z is even.

For $0 \leq k \leq 5$,
 $x + y = 10 - 2k$

has exactly $(10 - 2k) + 1$ non-negative integral solutions

\therefore number of ways in which z is even is

$$\sum (11 - 2k) = 6^2 = 36.$$

Thus, probability of required event

$$= \frac{36}{66} = \frac{6}{11}$$

3. Equation of any plane through $(1, 1, 1)$ is

$$a(x - 1) + b(y - 1) + c(z - 1) = 0 \quad \dots(i)$$

As (i) is perpendicular to the given planes

$$2a + b - 2c = 0$$

$$3a - 6b - 2c = 0$$

$$\Rightarrow \frac{a}{-2-12} = \frac{-b}{-4+6} = \frac{c}{-12-3}$$

or $\frac{a}{14} = \frac{b}{2} = \frac{c}{15}$

Thus, equation of required plane is

$$14(x - 1) + 2(y - 1) + 15(z - 1) = 0$$

or $14x + 2y + 15z = 31$

4. Write the differential equation as

$$\frac{dy}{dx} = \frac{1}{8\sqrt{x}\sqrt{9+\sqrt{x}}\sqrt{4+\sqrt{9+\sqrt{x}}}} = f(x) \text{ (say)}$$

$$\Rightarrow y = \int f(x) dx$$

Put $\sqrt{4 + \sqrt{9 + \sqrt{x}}} = t$

$$\Rightarrow \sqrt{9 + \sqrt{x}} = t^2 - 4$$

$$\Rightarrow 2t dt = \frac{1}{2\sqrt{9 + \sqrt{x}}} \cdot \frac{1}{2\sqrt{x}} dx$$

$$\therefore y = \int dt = t + C$$

$$= \sqrt{4 + \sqrt{9 + \sqrt{x}}} + C$$

Now, $\sqrt{7} = y(0) = \sqrt{4 + \sqrt{9 + 0}} + C$

$$\Rightarrow C = 0.$$

$$\therefore y(256) = \sqrt{4 + \sqrt{9 + \sqrt{256}}}$$

$$= \sqrt{4 + \sqrt{9 + 16}}$$

$$= \sqrt{4 + 5} = 3$$

5. As $f''(x) > 0 \forall x \in \mathbf{R}$,
 f' is a strictly increasing function on \mathbf{R} . By the first mean value theorem there exists some $\alpha \in (1/2, 1)$ such that
- $$\frac{f(1) - f(1/2)}{1 - 1/2} = f'(\alpha)$$
- $$\Rightarrow f'(\alpha) = 1 \text{ for some } \alpha \in (1/2, 1)$$
- As f' is strictly increasing on \mathbf{R} ,
- $$f'(x) > f'(\alpha) \forall x > \alpha$$
- $$\Rightarrow f'(1) > 1 \quad [\because 1 > \alpha]$$

6. $N_k = {}^5C_k + {}^4C_{5-k}$

$$\Rightarrow N_1 + N_2 + N_3 + N_4 + N_5$$

$$= (5)(1) + (10)(4) + (10)(6) + (5)(4) + (1)(1)$$

$$= 126$$

7. Let $\mathbf{OP} = \mathbf{p}$, $\mathbf{OQ} = \mathbf{q}$, $\mathbf{OR} = \mathbf{r}$, $\mathbf{OS} = \mathbf{s}$
 We are given

$$\mathbf{p} \cdot \mathbf{q} + \mathbf{r} \cdot \mathbf{s} = \mathbf{r} \cdot \mathbf{p} + \mathbf{q} \cdot \mathbf{s} = \mathbf{q} \cdot \mathbf{r} + \mathbf{p} \cdot \mathbf{s}$$

From first two expressions,

$$(\mathbf{r} - \mathbf{q}) \cdot \mathbf{p} + (\mathbf{q} - \mathbf{r}) \cdot \mathbf{s} = 0$$

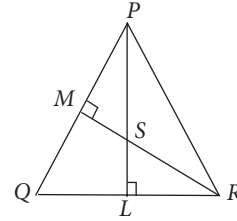
$$\Rightarrow (\mathbf{r} - \mathbf{q}) \cdot (\mathbf{p} - \mathbf{s}) = 0$$

$$\Rightarrow (\mathbf{QR}) \cdot (\mathbf{SP}) = 0 \Rightarrow \mathbf{SP} \perp \mathbf{QR} \quad \dots(1)$$

Similarly, from last two expressions,

$$\Rightarrow (\mathbf{PQ}) \cdot (\mathbf{SR}) = 0 \Rightarrow \mathbf{SR} \perp \mathbf{PQ} \quad \dots(2)$$

From (1), (2) we get S is orthocentre of ΔPQR .
 See figure in the next column.



8. For $k < x < k + 1$, $k \in \mathbf{N}$

$$\frac{1}{x+1} < \frac{1}{k+1} < \frac{1}{x} < \frac{1}{k}$$

$$\Rightarrow \frac{k+1}{x+1} < 1 \Rightarrow \frac{k+1}{(x+1)x} < \frac{1}{x}$$

$$\Rightarrow \int_k^{k+1} \frac{k+1}{(x+1)x} dx < \int_k^{k+1} \frac{1}{x} dx = \ln(k+1) - \ln k$$

$$\Rightarrow \sum_{k=1}^{98} \int_k^{k+1} \frac{k+1}{(x+1)x} dx < \ln(99) - \ln(1)$$

$$= \ln(99)$$

Next, for $k < x < k + 1$, $1 \leq k \leq 98$

$$\frac{k+1}{x(x+1)} > \frac{k+1}{(k+1)(x+1)} = \frac{1}{x+1} > \frac{1}{100}$$

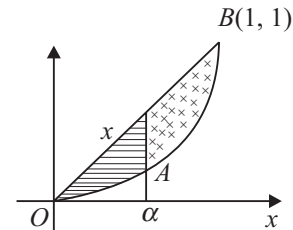
$$\Rightarrow \int_k^{k+1} \frac{k+1}{x(x+1)} dx > \frac{1}{100}$$

$$\Rightarrow \sum_{k=1}^{98} \int_k^{k+1} \frac{k+1}{x(x+1)} dx > \frac{98}{100} = \frac{49}{50}$$

9. Let $A_1 = \int_0^\alpha (x - x^3) dx$

$$A_2 = \int_\alpha^1 (x - x^3) dx$$

Now, $A_1 = A_2$



and $A_1 + A_2 = \int_0^1 (x - x^3) dx = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

$$\therefore 2A_1 = \frac{1}{4} \Rightarrow A_1 = \frac{1}{8}$$

$$\Rightarrow \int_0^\alpha (x - x^3) dx = \frac{1}{8}$$

$$\Rightarrow \frac{1}{2} \alpha^2 - \frac{1}{4} \alpha^4 = \frac{1}{8}$$

$$\Rightarrow 4\alpha^2 - 2\alpha^4 = 1$$

$$\Rightarrow 2\alpha^4 - 4\alpha^2 + 1 = 0$$

Let $f(x) = 2x^4 - 4x^2 + 1$

We have

$$f\left(\frac{1}{2}\right) = 2\left(\frac{1}{16}\right) - 4\left(\frac{1}{4}\right) + 1 = \frac{1}{8} > 0$$

and $f(1) = 2 - 4 + 1 = -1 < 0$

$\Rightarrow f(x) = 0$ has a root in $(1/2, 1)$

We have $f'(x) = 8x^3 - 8x$

$= 8x(x^2 - 1) < 0$ for $0 < x < 1/2$

$\Rightarrow f(x)$ decrease in $(0, 1/2]$

Since $f(1/2) = 1/4 > 0$

$\Rightarrow f(x) > 0$ for $0 < x \leq 1/2$.

10. $g'(x) = 2 \cos(2x) \sin^{-1}(\sin 2x) - \cos x \sin^{-1}(\sin x)$

$g'\left(-\frac{\pi}{2}\right) = 0 \neq 2\pi, -2\pi$

and $g'\left(\frac{\pi}{2}\right) = 0 \neq 2\pi, -2\pi$

11. Using $C_1 \rightarrow C_1 - C_2$, we get

$$f(x) = \begin{vmatrix} 0 & \cos(2x) & \sin(2x) \\ -2\cos x & \cos x & -\sin x \\ 0 & \sin x & \cos x \end{vmatrix}$$

$= 2\cos x (\cos 2x \cos x - \sin x \sin 2x)$

$= 2\cos x \cos 3x = \cos 2x + \cos 4x$

$f(x)$ attains its maximum value 2 at $x = 0$

$f'(x) = -2 \sin 2x - 4\sin 4x$

$f'\left(-\frac{\pi}{2}\right) = 0, f'(0) = 0, f'\left(\frac{\pi}{2}\right) = 0$

12. $2(\cos \beta - \cos \alpha) + \cos \alpha \cos \beta = 1$

$\Rightarrow (\cos \alpha + 2)(\cos \beta - 2) = -3$

$\Rightarrow 2 + \cos \alpha = \frac{3}{2 - \cos \beta}$ (1)

Let $a = \tan(\alpha/2)$ and $b = \tan(\beta/2)$, now (1) can be written as

$$2 + \frac{1-a^2}{1+a^2} = \frac{3(1+b^2)}{2(1+b^2)-(1-b^2)}$$

$\Rightarrow \frac{3+a^2}{1+a^2} = \frac{3(1+b^2)}{1+3b^2}$

$\Rightarrow (3+a^2)(1+3b^2) = 3(1+a^2)(1+b^2)$

$\Rightarrow 3 + (a^2 + 9b^2) + 3a^2b^2$

$= 3 + 3a^2 + 3b^2 + 3a^2b^2$

$\Rightarrow a^2 = 3b^2 \Rightarrow a = \pm\sqrt{3}b$

$\Rightarrow \tan(\alpha/2) + \sqrt{3} \tan(\beta/2) = 0$

and $\tan(\alpha/2) - \sqrt{3} \tan(\beta/2) = 0$

13. $f(x) = \frac{1-x(1+|1-x|)}{|1-x|} \cos\left(\frac{1}{1-x}\right), x \neq 1$

$$= \begin{cases} \frac{1-x(1+(1-x))}{1-x} \cos\left(\frac{1}{1-x}\right) & \text{if } x < 1 \\ \frac{1-x(1+x-1)}{x-1} \cos\left(\frac{1}{1-x}\right) & \text{if } x > 1 \end{cases}$$

$$= \begin{cases} \frac{(1-x)^2}{1-x} \cos\left(\frac{1}{1-x}\right) & \text{if } x < 1 \\ \frac{1-x^2}{x-1} \cos\left(\frac{1}{1-x}\right) & \text{if } x > 1 \end{cases}$$

$$= \begin{cases} (1-x) \cos\left(\frac{1}{1-x}\right) & \text{if } x < 1 \\ -(x+1) \cos\left(\frac{1}{1-x}\right) & \text{if } x > 1 \end{cases}$$

For $x < 1, -1 \leq \cos\left(\frac{1}{1-x}\right) \leq 1$

$\Rightarrow -(1-x) \leq (1-x) \cos\left(\frac{1}{1-x}\right) \leq (1-x)$

Since $\lim_{x \rightarrow 1^-} (1-x) = 0$, we get

$\lim_{x \rightarrow 1^-} (1-x) \cos\left(\frac{1}{1-x}\right) = 0$

or $\lim_{x \rightarrow 1^-} f(x) = 0$

As $\lim_{x \rightarrow 1^+} \cos\left(\frac{1}{1-x}\right)$ doesn't exist,

$\lim_{x \rightarrow 1^+} (-x-1) \cos\left(\frac{1}{1-x}\right)$ doesn't exist

14. $f'(x) > 2f(x) \forall x \in \mathbf{R}$

Let $g(x) = e^{-2x} f(x)$, then

$g'(x) = e^{-2x} f'(x) - 2e^{-2x} f(x)$

$= e^{-2x} (f'(x) - 2f(x)) > 0 \forall x \in \mathbf{R}$

$\Rightarrow g$ is strictly increasing on \mathbf{R}

Also, $g(0) = f(0) = 1$

$\Rightarrow g(x) > g(0) \forall x > 0$

$\Rightarrow e^{-2x} f(x) > 1 \forall x > 0$

$\Rightarrow f(x) > e^{2x} \forall x > 0$

As $f'(x) > 2f(x) > e^{2x} > 0 \forall x > 0$

$\Rightarrow f(x)$ is increasing on $(0, \infty)$

Also, $f'(x) > 2f(x) > 2e^{2x} > e^{2x} \forall x \in (0, \infty)$

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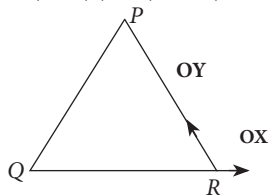
15. Let $E = \cos (P + Q) + \cos (Q + R) + \cos (R + P)$
 $= \cos (\pi - R) + \cos (\pi - P) + \cos (\pi - Q)$
 $= -(\cos P + \cos Q + \cos R)$

But in ΔPQR ,

$$\cos P + \cos Q + \cos R \leq 3/2$$

therefore $E \geq -3/2$

16. $|\mathbf{OX} \times \mathbf{OY}| = |\mathbf{OX}| |\mathbf{OY}| \sin (\pi - R)$



$$= \sin (P + Q)$$

17. $\alpha + \beta = 1, \alpha\beta = -1,$

$$\alpha = \frac{1}{2} (\sqrt{5} + 1), \beta = \frac{1}{2} (-\sqrt{5} + 1)$$

$$a_n = p\alpha^n + q\beta^n \quad \text{for } n = 0, 1, 2, 3, \dots$$

For $n \geq 1$

$$\begin{aligned} a_{n+1} &= p\alpha^{n+1} + q\beta^{n+1} \\ &= (\alpha + \beta) (p\alpha^{n+1} + q\beta^{n+1}) \\ &\quad - \alpha\beta (p\alpha^{n-1} + q\beta^{n-1}) \end{aligned}$$

$$\Rightarrow a_{n+1} = a_n + a_{n-1} \quad \forall n \geq 1 \quad (1)$$

$$a_0 = p + q$$

$$a_1 = p\alpha + q\beta$$

$$a_2 = a_1 + a_0$$

$$a_3 = a_1 + a_2$$

$$a_4 = a_2 + a_3 = a_2 + (a_1 + a_2)$$

$$= a_1 + 2a_2$$

$$= a_1 + 2(a_1 + a_0)$$

$$= 3a_1 + 2a_0$$

$$\Rightarrow 28 = 3(p\alpha + q\beta) + 2(p + q)$$

$$= \frac{7}{2}(p + q) + \frac{3\sqrt{5}}{2}(p - q)$$

As p, q are integers, we get

$$28 = \frac{7}{2}(p + q), p - q = 0$$

$$\Rightarrow p + q = 8, p = q \Rightarrow p = q = 4$$

$$\therefore p + 2q = 12$$

18. From (1)

$$a_{12} = a_{11} + a_{10}$$