Model Test Paper-I

SECTION I

(Single Correct Answer Type)

Each question has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

1. The tangent at any point P of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ makes an intercept p between the point of contact and the transverse axis of the hyperbola. p_1, p_2 are the lengths of the perpendiculars drawn from the focii on the normal at P, then p is

(a) an arithmetic mean between p_1 and p_2

(b) a geometric mean between p_1 and p_2

(c) a harmonic mean between p_1 and p_2

(d) none of these.

2. The distance of the origin from the image of (1, 2, 3) in the plane $x - y + z = 5$ is

(a) $\sqrt{14}$ (b) $\sqrt{2}$

(c) $\sqrt{34}$ (d) $\sqrt{41}$

3. Equation of the circle of minimum radius which touches both the parabolas $y = x^2 + 2x + 4$ and $x = y^2 + 2y + 4$ is

(a) $4x^2 + 4y^2 - 11x - 11y - 31 = 0$

(b) $4x^2 + 4y^2 + 11x - 11y + 13 = 0$

(c) $4x^2 + 4y^2 - 11x - 11y - 13 = 0$

(d) $4x^2 + 4y^2 - 11x - 11y + 13 = 0$

4. Let $z = \frac{1}{2}(\sqrt{3}-i)$, then $(i^{103}+z^{103})^{101}$ equals

(a) $\bar{z}$ (b) $i\bar{z}$

(c) z (d) $-iz$

5. The value of

$^{404}C_4 - {}^4C_1\,{}^{303}C_4 + {}^4C_2\,{}^{202}C_4 - {}^4C_3\,{}^{101}C_4$ is

(a) 101^4 (b) 404^2

(c) 202^4 (d) 303^2

6. Let A and B be two 2×2 matrices with entries from the set of integers such that $A^3 + B^3 = C$, where $C = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}$ then one of the possible value of $A + B$ is

(a) C (b) $-C$

(c) $I - C$ (d) $I + C$

7. The value of $\int \frac{-x}{x+1-\sqrt{x+1}}\,dx$ is

(a) $-x\sqrt{x+1}-x+C$

(b) $x-x\sqrt{x+1}+C$

(c) $-x-2\sqrt{x+1}+C$

(d) $2x+\sqrt{x+1}+C$

8. A bacteria culture grows with constant relative growth rate i.e. $y = Ce^{Kt}$. The bacteria count was 400 after 2 hours and 25,600 after 6 hours. The rate of growth after 4.5 hours is

(a) $25(\log 8)\ 8^{3/4}$

(b) $25(\log 6)\ 6^{5/4}$

(c) $25(\log 16)\ 8^{17/4}$

(d) $25(\log 8)\ 8^{9/4}$

9. Suppose that f is defined in a neighbourhood of x and $f''(x)$ exits. The value of

$$\lim_{h\to 0}\frac{f(x+h)+f(x-h)-2f(x)}{h^2} \text{ is}$$

(a) $f''(x) + f'(x)$ (b) $f''(x) + 2f'(x)$

(c) $f''(x)$ (d) $2f''(x)$

10. Let $f(x) = \begin{cases} x\sin(\pi/x) & \text{if } x>0 \\ 0 & \text{if } x=0 \end{cases}$

then

(a) $f'(x)$ exists at $x = 0$
(b) $f'(x)$ vanishes only at one point in (0, 1)
(c) $f'(x)$ vanishes only at two points in (0, 1)
(d) $f'(x)$ vanishes at infinitely many points in (0, 1)

SECTION II
(Multiple Correct Answer Type)

Each question has four choices (a), (b), (c) and (d) out of which **ONE or MORE** are correct.

11. A random variable X takes values –1, 0, 1, 2 with probabilities $\frac{1}{4}(1+3p), \frac{1}{4}(1-p), \frac{1}{4}(1+2p), \frac{1}{4}-p$ respectively, where p lies in a suitable interval I of **R**. Then

(a) $\underset{p\in I}{\text{Max}}\,(E(x)) < \frac{3}{2}$ (b) $\underset{p\in I}{\text{Min}}\,(E(x)) > -\frac{1}{8}$

(c) $-\frac{1}{16} \le E(x) \le \frac{5}{4}$ (d) $0 \le E(x) \le \frac{3}{2}$

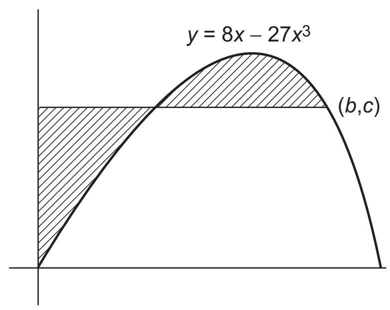
12. C_1 is a circle described on the line joining the positive ends of the major axis and minor axis of the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$ and C_2 is the circle described on the line joining the negative ends of the major axis and minor axis of the ellipse. If $y = mx + C$ is a common tangent to C_1 and C_2, then

(a) $m=-\frac{5}{4}, C=0$ (b) $m=\frac{4}{5}, C=0$

(c) $m=\frac{4}{5}, C=\frac{41}{10}$ (d) $m=\frac{4}{5}, C=\frac{-41}{10}$

13. Suppose $f'(x)$ exits for each x and $h(x) = 5f(x) - 3(f(x))^2 + (f(x))^3$, $x \in \mathbf{R}$, Then
(a) h increases whenever f decreases
(b) h increases whenever f increases
(c) h decreases whenever f decreases
(d) h increases or decreases according as f increases or decreases respectively.

14. The figure shows a horizontal line $y = c$ passing through (b, c) intersecting the curve $y = 8x - 27x^3$. If the shaded areas are equal then



(a) $b=\frac{1}{9}$ (b) $b=\frac{4}{9}$

(c) $c=\frac{32}{27}$ (d) $c=\frac{5}{27}$

15. Let $y = \frac{(25)(5^{2x})-(10)(5^x)+4}{(25)(5^{2x})+(10)(5^x)+4}$. Then
(a) Minimum value of y is 1/2
(b) Maximum value of y is 3
(c) Minimum value of y is 1/3
(d) y has no maximum value

SECTION III
(Integer Answer Type)

Answer to each question is a **single digit integer,** ranging from 0 to 9 (both inclusive).

16. Let $a_1, a_2, \ldots, a_8, > 1$. The minimum value of $E = \log_{a_1^2}(a_2)+\log_{a_2^2}(a_3)+\cdots+\log_{a_7^2}(a_8)+\log_{a_8^2}(a_1)$ is

17. If the lengths of latus rectums of the parabolas $y^2 = \lambda x$ and $25((x-3)^2 + (y+2)^2) = (3x - 4y - 2)^2$ are equal, then λ is equal to.

18. Suppose that a curve C passes through the point (3, 2) and has the property that if the normal line is drawn at any point on the curve then the y-intercept of the normal line is always 6. The curve C is a circle with radius

19. Suppose that f satisfies $f(x + y) = f(x) + f(y) + x^2y + xy^2$ for all real numbers x and y. If $\lim_{x\to 0}\frac{f(x)}{x}=2$ then $f'(2)$ is equal to

20. If [**c d a**] = 4 and (**b** × **c**) × (**d** × **a**) + (**b** × **d**) × (**a** × **c**) + (**b** × **a**) × (**c** × **d**) + k **b** = 0 then k is equal to

Answers

Section I

Single Correct Answer Type

1. (c) **2.** (c) **3.** (c)
4. (c) **5.** (a) **6.** (a)
7. (c) **8.** (d) **9.** (c)
10. (d)

Section II

Multiple Correct Answer Type

11. (a), (b), (c) **12.** (a), (c), (d) **13.** (b), (c), (d)
14. (b), (c) **15.** (c), (d)

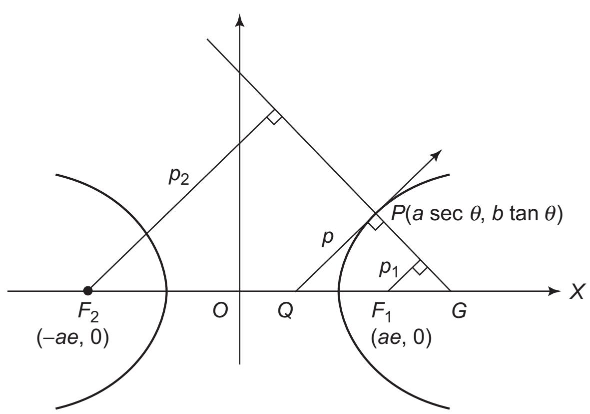
Section III

Integer Answer Type

16. 4 **17.** 6 **18.** 5
19. 6 **20.** 8

Hints and Solutions

1. Equation of the tangent at $P(a \sec\theta, b\tan\theta)$ is $\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1$ which meets the transverse axis $y = 0$ at Q $(a\cos\theta, 0)$. $F_1(ae, 0)$ and $F_2(-ae, 0)$ be the two focii



Equation of the normal at P is

$$\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2e^2.$$

which meets the axis $y = 0$ at G $(ae^2\sec\theta, 0)$

From Δ GPQ,

$$\frac{p}{p_1} = \frac{QG}{F_1G} = \frac{ae^2\sec\theta - a\cos\theta}{ae^2\sec\theta - ae}$$

$$= \frac{e^2 - \cos^2\theta}{e^2 - e\cos\theta} = \frac{e + \cos\theta}{e} = 1 + \frac{\cos\theta}{e}$$

Similarly $\frac{p}{p_2} = 1 - \frac{\cos\theta}{e}$.

so that $\frac{p}{p_1} + \frac{p}{p_2} = 2 \Rightarrow \frac{1}{p_1} + \frac{1}{p_2} = \frac{2}{p}$.

$\Rightarrow p$ is the harmonic mean between p_1 and p_2

2. Let Q be the image of the point $P(1, 2, 3)$ in the plane $x - y + z = 5$. PQ is perpendicular to the plane and the midpoint R of PQ lies on the plane

Equation of PQ is $\frac{x-1}{1} = \frac{y-2}{-1} = \frac{z-3}{1}$

and point Q be $(r + 1, -r + 2, r + 3)$

Coordinates of R are $\left(\frac{r+2}{2}, \frac{-r+4}{2}, \frac{r+6}{2}\right)$

As R lies on the plane,

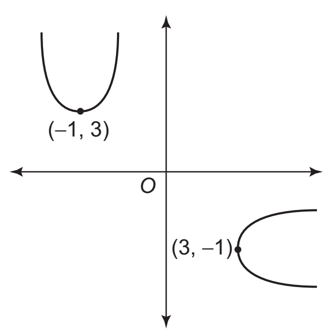
$$\Rightarrow \frac{r+2}{2} - \frac{4-r}{2} + \frac{r+6}{2} = 5 \Rightarrow 3r = 6 \Rightarrow r = 2$$

So image of P is Q (3, 0, 5) whose distance from the origin is $\sqrt{3^2 + 0^2 + 5^2} = \sqrt{34}$

3. Equations of the given parabolas can be written as $P_1 : y - 3 = (x + 1)^2$ and $P_2 : (x - 3) = (y + 1)^2$

The parabolas are symmetric about the line $y = x$. So they have a common normal with slope, –1

which meets P_1 at $y - 3 = \frac{1}{4}(-1)^2$ and $x + 1 = -2 \times \frac{1}{4}(-1)$ i.e. $\left(-\frac{1}{2}, \frac{13}{4}\right)$



(Using, the normal with slope m to the parabola $y^2 = 4ax$ meets the curve at $(am^2, -2am)$)

and P_2 at $\left(\frac{13}{4}, -\frac{1}{2}\right)$

Circle with minimum radius touching both the curves is along the common normal so its equation is

$$\left(x - \frac{13}{4}\right)\left(x + \frac{1}{2}\right) + \left(y + \frac{1}{2}\right)\left(x - \frac{13}{4}\right) = 0$$

$\Rightarrow \quad x^2 + y^2 - \frac{11}{4}x - \frac{11}{4}y - \frac{13}{4} = 0$

$\Rightarrow \quad 4x^2 + 4y^2 - 11x - 11y - 13 = 0$

4. Write

$$z = \cos\left(\frac{\pi}{6}\right) - i\sin\left(\frac{\pi}{6}\right)$$

$$z^{103} = \cos\left(\frac{103\pi}{6}\right) - i\sin\left(\frac{103\pi}{6}\right)$$

$$= \cos\left(18\pi - \frac{5\pi}{6}\right) - i\sin\left(18\pi - \frac{5\pi}{6}\right)$$

$$= \cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)$$

$$= -\cos\left(\frac{\pi}{6}\right) + \frac{i}{2}$$

Also, $\quad i^{103} = (i^4)^{25}\, i^3 = -i$

$$\therefore\ i^{103} + z^{103} = -\cos\left(\frac{\pi}{6}\right) - \frac{i}{2}$$

$$= -\left[\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right]$$

$$\Rightarrow (i^{103} + z^{103})^{101} = -\left[\cos\left(\frac{101\pi}{6}\right) + i\sin\left(\frac{101\pi}{6}\right)\right]$$

$$= -\left[\cos\left(16\pi + \frac{5\pi}{6}\right) + i\sin\left(16\pi + \frac{5\pi}{6}\right)\right]$$

$$= -\left[\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right]$$

$$= -\left[-\frac{\sqrt{3}}{2} + \frac{i}{2}\right] = z$$

5. ${}^{404}C_4 - {}^4C_1\ {}^{303}C_4 + {}^4C_2\ {}^{202}C_4 - {}^4C_3\ {}^{101}C_4$

= coefficient of x^4 in

$(1 + x)^{404} - {}^4C_1\ (1 + x)^{303} + {}^4C_2\ (1 + x)^{202} - {}^4C_3\ (1 + x)^{101}$

= coefficient of x^4 in $((1 + x)^{101} - 1)^4$

= coefficient of x^4 in $({}^{101}C_1 x + {}^{101}C_2 x^2 + \ldots)^4$

= $(101)^4$

6. Write $C = -I + D$ where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$.

$\Rightarrow \quad C + I = D$

$\Rightarrow \quad (C + I)^2 = D^2 = -D$

$\Rightarrow \quad C^2 + 2C + I = -(C + I)$

$\Rightarrow \quad C^2 + 3C + 2I = 0$

$\Rightarrow \quad C^3 + 3C^2 + 2C = 0$

$\Rightarrow \quad (C + I)^3 + (-I)^3 = C$

Let $C + I = A$, $B = -I$, so that $A + B = C$.

7. Let $u = \sqrt{x+1}$ i.e. $u^2 = x + 1 \Rightarrow x = u^2 - 1$ so $dx = 2u\ du$

$$I = \int \frac{-x}{x + 1 - \sqrt{x+1}}\, dx = \int \frac{1 - u^2}{u^2 - u}\, 2u\, du$$

$$= -2\int (1 + u)\, du$$

$$= -2\left(u + \frac{u^2}{2}\right) + C$$

$$= -2u - u^2 + C$$

$$= -2\sqrt{x+1} - (x + 1) + C$$

$$= -x - 2\sqrt{x+1} + C$$

8. We need to determine constants k and C. According to the given condition

$$400 = C\, e^{2K}$$

$\Rightarrow \quad \log(400) = \log C + 2K$

$$\Rightarrow \quad K = \frac{1}{2}[\log 400 - \log C]$$

$$= \log\left(\frac{20}{\sqrt{C}}\right)$$

Also $\quad 25{,}600 = C\, e^{6\log(20/\sqrt{C})}$

$$= C\left(\frac{20}{\sqrt{C}}\right)^6 = \frac{20^6}{C^2}$$

$$\Rightarrow \quad C^2 = \frac{20^6}{2^8 \times 10^2} = \frac{10^4}{2^2}$$

$$C = 50, \text{ and } K = \frac{1}{2}\log 8$$

Thus $\quad y = 50\, e^{\frac{1}{2}(\log 8)t}$

$$\frac{dy}{dt} = 50\left(\frac{1}{2}\log 8\right)e^{\frac{1}{2}(\log 8)t}$$

Setting $\quad t = 4.5$, we have

$$\frac{dy}{dt} = 25 \log 8\ e^{\frac{9}{4}(\log 8)}$$
$$= 25 (\log 8)\ 8^{9/4}.$$

9. $\lim_{h\to 0} \frac{f(x+h)+f(x-h)-2f(x)}{h^2}$

$$= \lim_{h\to 0} \frac{f'(x+h)-f'(x-h)}{2h} \qquad \left(\frac{0}{0}\text{ form}\right)$$

$$= \frac{1}{2}\lim_{h\to 0}\left[\frac{f'(x+h)-f'(x)}{h}+\frac{f'(x-h)-f'(x)}{-h}\right]$$

$$= \frac{1}{2}[f''(x)+f''(x)] = f''(x)$$

10. $\lim_{h\to 0+}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0+}\sin\left(\frac{\pi}{h}\right)$ which does not exist. For $x > 0$

$$f'(x) = \sin\left(\frac{\pi}{x}\right)-\frac{\pi}{x}\cos\left(\frac{\pi}{x}\right)$$

Consider $I_n = \left[\frac{1}{2n}, \frac{1}{2(n+1)}\right] \subset (0,1)$

f is continuous on I_n and differentiable on $\left(\frac{1}{2n}, \frac{1}{2(n+1)}\right)$ also $f\left(\frac{1}{2n}\right) = \frac{1}{2n}\sin(2n\pi) = 0$ and $f\left(\frac{1}{2(n+1)}\right) = 0$. By the Rolle's theorem there is $C_n \in \left(\frac{1}{2n}, \frac{1}{2(n+1)}\right)$ such that $f'(C_n) = 0$. Hence f' vanishes at $C_1, C_2, C_3 \ldots$.

11. We must have

$$\frac{1}{4}(1+3p)\ge 0, \frac{1}{4}(1-p)\ge 0, \frac{1}{4}(1+2p)\ge 0,$$
$$\frac{1}{4}-p\ge 0$$

and $\frac{1}{4}(1+3p)+\frac{1}{4}(1-p)+\frac{1}{4}(1+2p)+\frac{1}{4}(1-4p) = 1$

$$\Rightarrow \quad p\ge -\frac{1}{3}, p\le 1, p\ge -\frac{1}{2}, p\le \frac{1}{4}$$

$$\Rightarrow \quad -\frac{1}{3}\le p\le \frac{1}{4} \qquad (1)$$

Now, note that $I = \left[\frac{-1}{3}, \frac{1}{4}\right]$.

$$E(X) = (-1)\frac{1}{4}(1+3p)+(0)\frac{1}{4}(1-p)+(1)\frac{1}{4}(1+2p) + (2)\left(\frac{1}{4}-p\right)$$

$$= \frac{1}{4}(2-9p)$$

From (1)

$$2-\frac{9}{4}\le 2-9p\le 2+3$$

$$\Rightarrow \quad -\frac{1}{16}\le E(X)\le \frac{5}{4}$$

Thus,

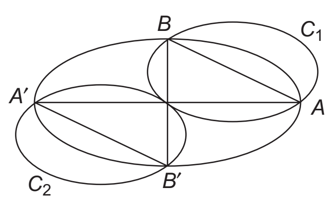
$$\max_{p\in I} E(X) = \frac{5}{4} < \frac{3}{2} \quad \min_{p\in I} E(X) = -\frac{1}{16} > -\frac{1}{8}.$$

12. Ends of the major axis are $(\pm 5, 0)$ and of minor axis are $(0, \pm 4)$. So equations of C_1 and C_2 are.

$$C_1 : \left(x-\frac{5}{2}\right)^2+(y-2)^2 = \left(\frac{5}{2}\right)^2+2^2$$

$\Rightarrow \quad x^2 + y^2 - 5x - 4y = 0$

and $C_2 : x^2 + y^2 + 5x + 4y = 0$



Two circlers touch each other externally at the origin. So equation of the common tangent at the origin is $5x + 4y = 0$ for which $m = -5/4$, $C = 0$. Next, other two common tangents will be parallel to the line joining their centres $\left(\frac{5}{2}, 2\right)$ and $\left(-\frac{5}{2}, -2\right)$, at a distance $\sqrt{\left(\frac{5}{2}\right)^2+2^2}$ equal to the radius of the two circles. From their centres.

$$\left|\frac{2-(4/5)(5/2)-C}{\sqrt{1+(4/5)^2}}\right| = \sqrt{\left(\frac{5}{2}\right)^2+2^2}$$

$$\Rightarrow \quad C = \frac{\pm 41}{10}$$

13. $h'(x) = 5f'(x) - 6f(x)f'(x) + 3(f(x))^2 f'(x)$

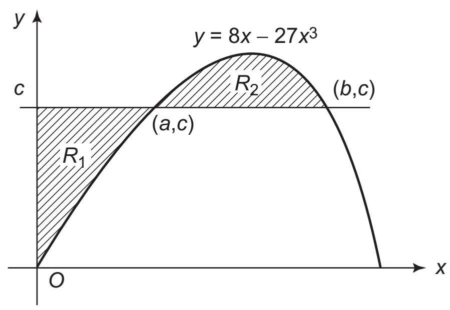
$= f'(x)\,[3(f(x))^2 - 6f(x) + 5]$

$$= 3f'(x)\left[(f(x))^2 - 2f(x) + \frac{5}{3}\right]$$

$$= 3f'(x)\left[(f(x)-1)^2 + \frac{2}{3}\right]$$

Thus $h'(x) > 0$ or < 0 according as $f'(x) > 0$ or < 0 respectively.

14. Let a and b be the x-coordinates of the points where the line intersects the curve. From the figure



$$R_1 = R_2$$

$$\Rightarrow \int_0^a [c - (8x - 27x^3)]dx$$

$$= \int_a^b [8x - 27x^3 - c)]dx$$

$$\Rightarrow \left[cx - 4x^2 + \frac{27}{4}x^4\right]_0^a = \left[4x^2 - \frac{27}{4}x^4 - cx\right]_a^b$$

$$\Rightarrow ac - 4a^2 + \frac{27}{4}a^4 = 4b^2 - \frac{27}{4}b^4 - cb$$

$$-\left(4a^2 - \frac{27}{4}a^4 - ac\right)$$

$$\Rightarrow 0 = 4b^2 - \frac{27}{4}b^4 - cb$$

$$= 4b^2 - \frac{27}{4}b^4 - b(8b - 27b^3)$$

$$= b^2\left(\frac{81}{4}b^2 - 4\right)$$

Since $b > 0 \Rightarrow b = \frac{4}{9}$. Thus $c = 8b - 27b^3 = \frac{32}{27}$.

15. Let $t = 5^{x+1}$, so that

$$y = \frac{t^2 - 2t + 4}{t^2 + 2t + 4}$$

Note that $t > 0$ and $y \neq 1$.

$\Rightarrow \quad (t^2 + 2t + 4)y = t^2 - 2t + 4$

$\Rightarrow \quad (y - 1)t^2 + 2(y + 1)t + 4(y - 1) = 0.$

As t is real

$4(y + 1)^2 - 16(y - 1)^2 \geq 0.$

$\Rightarrow \quad (y + 1 + 2y - 2)\,(y + 1 - 2y + 2) \geq 0$

$$\Rightarrow \quad 3\left(y - \frac{1}{3}\right)(y - 3) \leq 0$$

$$\Rightarrow \quad \frac{1}{3} \leq y \leq 3.$$

Now, $y = \frac{1}{3} \Rightarrow -\frac{2}{3}t^2 + \frac{8}{3}t - \frac{8}{3} = 0$

$\Rightarrow \quad t^2 - 4t + 4 = 0 \Rightarrow t = 2.$

$\therefore \quad 5^{x+1} = 2 \Rightarrow x = \log_5 2 - 1$

Similarly $y = 3 \Rightarrow 2t^2 + 8t^2 + 8 = 0$

$\Rightarrow \quad (t + 2)^2 = 0 \Rightarrow t = -2.$

Thus, maximum value is never attained.

16. Using change of base formula, we get

$$E = \frac{\log(a_2)}{\log(a_1^2)} + \frac{\log(a_3)}{\log(a_2^2)} + \cdots + \frac{\log(a_8)}{\log(a_7^2)} + \frac{\log a_1}{\log(a_8^2)}$$

$$\Rightarrow \quad \frac{1}{4}E = \frac{1}{8}\left[\frac{\log a_2}{\log a_1} + \frac{\log a_3}{\log a_2} + \cdots + \frac{\log a_1}{\log a_8}\right]$$

$$\geq \left[\frac{\log a_2}{\log a_1} \cdot \frac{\log a_3}{\log a_2} \cdots\cdots \frac{\log a_1}{\log a_8}\right]^{1/8} = 1$$

$\Rightarrow \quad E \geq 4$

$\therefore$ Minimum value of E is 4.

17. Length of the latus rectum of $y^2 = \lambda x$ is λ. Equation of the other parabola can be written as

$$\frac{3x - 4y - 2}{\sqrt{3^2 + 4^2}} = \sqrt{(x - 3)^2 + (y + 2)^2}$$

So the coordinates of the focus are $(3, -2)$ and equation of the directrix is $3x - 4y - 2 = 0$. Latus rectum of the parabola is equal to twice the distance between the focus and the directrix.

$$\Rightarrow \quad \lambda = 2 \times \frac{9 + 8 - 2}{\sqrt{3^2 + 4^2}} = 6.$$

18. Let $P(a, b)$ be any point on the curve. If m is the slope of the tangent line at P then the equation of the normal will be

$$Y - b = -\frac{1}{m}(X - a)$$

$$\Rightarrow \quad Y = -\frac{1}{m}X + \frac{a}{m} + b$$

The y-intercept is $\frac{a}{m} + b = 6$

$$\Rightarrow \quad m = \frac{a}{6-b}$$

So we solve $\frac{dy}{dx} = \frac{x}{6-y}$

$$\Rightarrow \quad 6y - \frac{y^2}{2} = \frac{x^2}{2} + C$$

$\Rightarrow \quad 12y - y^2 = x^2 + C$

Since (3, 2) is on the curve so $C = 11$

Thus $x^2 + y^2 - 12y + 11 = 0$

$\Rightarrow \quad x^2 + (y - 6)^2 = 25$

which is a circle with radius 5.

19. $$f'(2) = \lim_{h\to 0} \frac{f(2+h) - f(2)}{h}$$

$$= \lim_{h\to 0} \frac{f(2) + f(h) + 4h + 2h^2 - f(2)}{h}$$

$$= \lim_{h\to 0} \left[\frac{f(h)}{h} + 4 + 2h\right] = 2 + 4 = 6.$$

20. $(\mathbf{b} \times \mathbf{c}) \times (\mathbf{d} \times \mathbf{a}) = (\mathbf{b} \cdot (\mathbf{d} \times \mathbf{a}))\mathbf{c} - (\mathbf{c} \cdot (\mathbf{d} \times \mathbf{a}))\mathbf{b}$

$= [\mathbf{b}\ \mathbf{d}\ \mathbf{a}]\ \mathbf{c} - [\mathbf{c}\ \mathbf{d}\ \mathbf{a}]\ \mathbf{b}$ (i)

Also

$(\mathbf{b} \times \mathbf{a}) \times (\mathbf{c} \times \mathbf{d}) = [\mathbf{b}\ \mathbf{c}\ \mathbf{d}]\ \mathbf{a} - [\mathbf{a}\ \mathbf{c}\ \mathbf{d}]\mathbf{b}$ (ii)

and $(\mathbf{b} \times \mathbf{d}) \times (\mathbf{a} \times \mathbf{c})$

$= -(\mathbf{a} \times \mathbf{c}) \times (\mathbf{b} \times \mathbf{d})$

$= (\mathbf{a} \times \mathbf{c}) \times (\mathbf{d} \times \mathbf{b})$

$= [\mathbf{a}\ \mathbf{d}\ \mathbf{b}]\ \mathbf{c} - [\mathbf{c}\ \mathbf{d}\ \mathbf{b}]\ \mathbf{a}$

$= -[\mathbf{b}\ \mathbf{d}\ \mathbf{a}]\ \mathbf{c} - [\mathbf{c}\ \mathbf{d}\ \mathbf{b}]\ \mathbf{a}$ (iii)

Adding (i), (ii) and (iii), we have

$(\mathbf{b} \times \mathbf{c}) \times (\mathbf{d} \times \mathbf{a}) + (\mathbf{b} \times \mathbf{d}) \times (\mathbf{a} \times \mathbf{c}) + (\mathbf{b} \times \mathbf{a}) \times (\mathbf{c} \times \mathbf{d})$

$= -2[\mathbf{c}\ \mathbf{d}\ \mathbf{a}]\ \mathbf{b} = -8\mathbf{b}$