Model Test Paper-II

SECTION I

(Single Correct Answer Type)

Each question has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

1. $\sin^{-1}\left[\sin\left(\frac{2x^2+4}{1+x^2}\right)\right] < \pi - 3$ if

(a) $-1 \le x \le 0$ (b) $0 \le x \le 1$
(c) $-1 < x < 1$ (d) $x > 1$

2. The plane $2x - y + 3z + 5 = 0$ is rotated through 90° about its line of intersetion with the plane $5x - 4y - 2z + 1 = 0$. The equation of the plane in the new position is

(a) $26x - 9y - 29z - 13 = 0$
(b) $27x - 24y - 26z - 13 = 0$
(c) $27x - 24y + 26z - 13 = 0$
(d) $27x + 24y - 26z + 13 = 0$

3. 7 different coloured flags are to be hoisted on 10 ploes. More than one flag may be hoisted on a single pole. The probability that all the 7 flags are hoisted on different poles is

(a) $\frac{3}{143}$ (b) $\frac{3}{286}$
(c) $\frac{1}{143}$ (d) $\frac{1}{286}$

4. Suppose $a_i, b_i > 0$ for $i = 1, 2, ... n$.

Let $\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i = 10$, then least value of

$\sum_{i=1}^{n} \frac{a_i^2}{a_i + b_i}$ is

(a) 5 (b) 7.5
(c) 10 (d) 15

5. Let C and D be two square matrices of the same size such that $CD' + DC' = 0$. If C is a skew-symmetric matrix, then DC is

(a) a symmetric matrix
(b) a skew-symmetric matrix
(c) an orthgonal matrix
(d) an invertible matrix

6. Let $f: \left[\frac{1}{3}, 2\right] \to \mathbf{R}$ be a positive, non-constant and differentiable function such that $f'(x) < 3f(x)$ and $f(1/3) = 2$. Then the value of $\int_{1/3}^{2} f(x)\,dx$ lies in the interval

(a) $\left(\frac{2}{3}e - 1, \frac{2}{3}e\right)$ (b) $\left[0, \frac{2}{3}(e^5 - 1)\right)$
(c) $(e - 2, e + 2)$ (d) $\left[0, \frac{2}{3}(e^5 - 2)\right]$

7. Let $f(x) = \begin{cases} 4 & \text{for } x \ge 0 \\ 0 & \text{for } x < 0 \end{cases}$ and $g(x) = x^2$ for all $x \in \mathbf{R}$. Then

(a) f and g are continuous functions on **R**
(b) $f + g$ is continuous on **R**
(c) fg is continuous on **R**
(d) gof is continuous on **R**

8. A plane contains the points $P = (1, 0, 0)$, $Q = (1, 1, 1)$ and $R = (2, -1, 3)$. A unit vector orthogonal to the plane is

(a) $\frac{1}{\sqrt{6}}(-2\mathbf{i} + \mathbf{j} + \mathbf{k})$ (b) $\frac{1}{\sqrt{6}}(2\mathbf{i} + \mathbf{j} - \mathbf{k})$
(c) $\frac{1}{\sqrt{18}}(-4\mathbf{i} + \mathbf{j} + \mathbf{k})$ (d) $\frac{1}{\sqrt{18}}(4\mathbf{i} + \mathbf{j} - \mathbf{k})$

SECTION–II

(Paragraph Type)

Six multiple choice questions relating to **3 paragraphs with two questions on each paragraph**. Each question has four choices (a), (b), (c) and (d), out of which **ONLY ONE** is correct.

Paragraph for Questions Nos. 9 and 10

Let A = {–1, 0, 1}, and let M = $\{(a_{ij})_{3\times3} | a_{ij} \in A\}$

9. The number of symmetric matrices in M is

(a) 81 (b) 243
(c) 729 (d) 2343

10. The number of skew symmetric matrices in M is

(a) 27 (b) 81
(c) 243 (d) 781

Paragraph for Questions Nos. 11 and 12

One can verify that $y = x$ is solution of the equation $y' = x^3 (y-x)^2 + x^{-1} y$. We reduce this equation to an equation which is reducible to a linear equation by substitution $w = y - x$.

11. The solution of this differential is given by

(a) $-\dfrac{1}{(y-x)} x = \dfrac{x^5}{5} + C$

(b) $-\dfrac{1}{(y-x)^2} = \dfrac{x^6}{6} + C$

(c) $\dfrac{x}{(y-x)^2} = \dfrac{x^3}{3} + C$

(d) $\left(\dfrac{x}{y-x}\right)^2 = \dfrac{x^5}{5} + C$

12. Solution of the intial value problem satisfying $y(1) = 2$ and $y' = x^3 (y-x)^2 + x^{-1} y$ is given by

(a) $y = \dfrac{1}{5}(x^6 - 6)(x - y)$

(b) $y = \dfrac{1}{4}(x^5 - 5)(x - y)$

(c) $x = \dfrac{1}{5}(x^6 - 6)(x - y)$

(d) $x^2 = \dfrac{1}{5}(x^6 - 6)(x - y)$

Paragraph for Questions Nos. 13 and 14

H : $x^2 - y^2 = 9$, P : $y^2 = 4(x - 5)$ L : $x = 9$

13. If R is the point of intersection of the tangents to the rectangular hyperbola H at the extremities of the chord L, then equation of the chord of contact of R with respect to the parabola P is

(a) $x = 7$ (b) $x = 9$
(c) $y = 7$ (d) $y = 9$

14. If the chord of contact of R with respect to the parabola P meets the parabola at T and T', S is the focus of the parabola, then area of the triangle STT' is equal to

(a) 8 sq. units (b) 9 sq. units
(c) 12 sq. units (d) 16 sq. units

SECTION–III

(Multiple Correct Answer Type)

Each question has four choices (a), (b), (c) and (d), out of which **ONE** or **MORE** are correct.

15. In a triangle ABC, which of the following statements are true

(a) Maximum value of $\sin 2A + \sin 2B + \sin 2C$ is same as maximum value of $\sin A + \sin B + \sin C$.

(b) $R \geq 2r$, where R is the circumradius and r is the inradius of the triangle.

(c) $R^2 \geq \dfrac{abc}{a+b+c}$

(d) If s is the semiperimeter of the triangle then $r + 2R = s$, if the triangle is right angled.

16. Consider the lines $L_1 : \mathbf{r} = \mathbf{i} + \mathbf{j} + \lambda_1(\mathbf{i} - \mathbf{k})$ and $L_2 : \mathbf{r} = \mathbf{j} + \mathbf{k} + \lambda_2(\mathbf{i} + \mathbf{j})$. AB is the line of shortest distance between. L_1 and L_2.

(a) shortest distance between L_1 and L_2 is $\sqrt{3}/2$

(b) Position vector of the point of intersection of L_1 and L_2 is $\mathbf{j} + \mathbf{k}$

(c) shortest distance between L_1 and L_2 is zero

(d) Length of the projeetion of **OA** on $\mathbf{r} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ is $2/\sqrt{3}$

17. Let $f: [-1, 1] \to \mathbf{R}$ be defined by $f(x) = xe^{-x^2}$. Then

(a) f is strictly increasing on $[-1, 1]$

(b) f is strictly decreasing on $[0, 1]$

(c) f is strictly decreasing on $[1/\sqrt{2}, 1]$

(d) f is strictly increasing on $[0, \frac{1}{\sqrt{2}}]$

18. Let $I = \int(\sqrt{\tan x} - \sqrt{\cot x})\,dx$, then I equals

(a) $-\sqrt{2}\ \log|\sin x + \cos x + \sqrt{\sin 2x}| + C$

(b) $\sqrt{2}\ \log|\sin x - \cos x + \sqrt{\cos 2x}| + C$

(c) $\log\left|\frac{\tan^2 x - 1}{\sqrt{2\tan x}}\right| + C$

(d) $\log|\sqrt{\tan x} + \sqrt{\cot x} - \sqrt{\sin x}| + C$

19. Suppose $a, b, c, d, k \in \mathbf{R}$, be such that $|a|, |b|, |c|, |d|, \le k$, and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then

(a) $|A| \le (|a| + |b|)(|c| + |d|)$

(b) $|A| \le \max\{|a|, |b|\} \max\{|c|, |d|\}$

(c) $|A| \le \sqrt{a^2 + b^2}\sqrt{c^2 + d^2}$

(d) $|A| \le 2k^2$

20. Let $A_0 = -3$, $H_0 = -17$. For $n \ge 1$, let A_n, H_n denote the arithmetic and harmonic means respectively of A_{n-1} and H_{n-1}. Then

(a) $A_{20} < A_{21} < A_{22}$ (b) $H_4 > H_5 > H_6$

(c) $A_{99} < H_{101}$ (d) $A_{100}\ H_{100} = 51$

Answers

Section I
Single Correct Answer Type

1. (c) **2.** (b) **3.** (b)
4. (a) **5.** (b) **6.** (b)
7. (c) **8.** (d)

Section II
Paragraph Type

9. (c) **10.** (a) **11.** (a)
12. (c) **13.** (b) **14.** (c)

Section III
Multiple Correct Answer Type

15. (a), (b), (c), (d)
16. (b), (c), (d)
17. (c), (d)
18. (a)
19. (a), (c), (d)
20. (a), (c), (d)

Hints and Solutions

1. Let $t = \frac{2x^2 + 4}{1 + x^2} \Rightarrow x^2 = \frac{4 - t}{t - 2}$

Since $x^2 \ge 0$, $2 < t \le 4$

So, we have to solve $\sin^{-1}(\sin t) < \pi - 3$ and $2 < t \le 4$ simultaneously. $\frac{\pi}{2} < 2 < t \le 4 < 3\pi/2$

$\Rightarrow \sin^{-1}(\sin t) = \pi - t < \pi - 3 \Rightarrow t > 3$

$\Rightarrow 3 < t \le 4$

$\Rightarrow 3 < \frac{2x^2 + 4}{1 + x^2} \le 4 \Rightarrow x^2 < 1$

$\Rightarrow -1 < x < 1$

2. Equation of a plane passing through the line of intersection of these planes is $2x - y + 3z + 5 + \lambda(5x - 4y - 2z + 1) = 0$ or

$(2 + 5\lambda)x - (1 + 4\lambda)y + (3 - 2\lambda)z + 5 + \lambda = 0$

This will be perpendicular to the plane

$2x - y + 3z + 5 = 0$ if

$2(2 + 5\lambda) + (1 + 4\lambda) + 3(3 - 2\lambda) = 0$

$\Rightarrow \lambda = -7/4$ and the required equation of the plane is $27x - 24y - 26z - 13 = 0$

3. The number of ways hoisting first flag is 10; the second flag can be hoisted in (9+2) or 11 ways; the third flag can be hoisted in either (8 + 2 + 2) or (9+3) ways and so on.

Thus, 7 flags can be hoisted in (10)(11)(12)(13)(14)(15)(16) ways.

The flags can be hoisted on different poles in $({}^{10}C_7)(7!) = {}^{10}P_7$ ways

Thus, probability of the required event is

$$= \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4}{(10)(11)(12)(13)(14)(15)(16)} = \frac{3}{286}$$

4. We have

$$\frac{a_i^2}{a_i + b_i} = \frac{a_i(a_i + b_i) - a_i b_i}{a_i + b_i}$$

$$= a_i - \frac{a_i b_i}{a_i + b_i}$$

As H.M. $\le$ A.M., we get

$$\frac{2a_ib_i}{a_i+b_i} \le \frac{1}{2}(a_i+b_i)$$

$$\therefore \frac{a_i^2}{a_i+b_i} \ge a_i - \frac{1}{4}(a_i+b_i)$$

$$\Rightarrow \sum_{i=1}^{n}\frac{a_i^2}{a_i+b_i} \ge \sum_{i=1}^{n} a_i - \frac{1}{4}\sum_{i=1}^{n}(a_i+b_i)$$

$$\Rightarrow \sum_{i=1}^{n}\frac{a_i^2}{a_i+b_i} \ge 10 - \frac{1}{4}(10+10) = 5$$

The equality holds when $a_i = b_i$, for $1 \le i \le n$.

5. Let E = DC, then

E′ = (DC)′ = C′D′

= (–C)D′ = –CD′

= DC′ = D(–C)

= –DC = –E

⇒ E is a skew symmetric matrix.

6. Since $f(x) > 0$ so $\int_{1/3}^{2} f(x)dx \ge 0$

Also $f'(x) < 3f(x) \Rightarrow \log|f(x)| < 3x + C$

so $\log f(x) - \log f(1/3) < 3x - 1$

$\Rightarrow \log\frac{f(x)}{2} < 3x - 1$

$\Rightarrow f(x) < 2e^{3x-1}$

$$\Rightarrow \int_{1/3}^{2} f(x)dx < 2\int_{1/3}^{2} e^{3x-1}dx = \frac{2}{3}(e^5 - e^0)$$

$$= \frac{2}{3}(e^5 - 1)$$

7. Clearly f is not continuous at $x = 0$, since $\lim_{x\to 0^+} f(x) = 4$ and $\lim_{x\to 0^-} f(x) = 0$. Also

$$(f+g)(x) = \begin{cases} 4+x^2 & x \ge 0 \\ x^2 & x < 0 \end{cases}$$

so $f + g$ is not continuous at $x = 0$

$$(fg)(x) = f(x)\,g(x) = \begin{cases} 4x^2 & x \ge 0 \\ 0 & x < 0 \end{cases}$$

fg is a continuous function as $\lim_{x\to 0^+}(fg)(x) = 0 = \lim_{x\to 0^-}(fg)(x) = fg(0)$.

$$gof(x) = g(f(x)) = \begin{cases} 16 & x \ge 0 \\ 0 & x < 0 \end{cases}$$

gof is not continuous at $x = 0$

8. Since all three points lie in the plane the vectors **PQ** and **PR** lie in the plane.

PQ = **j** + **k** , **PR** = **i** – **j** + 3**k**

The cross product of the these two vectors is orthogonal to the plane

PQ × **PR** = 4**i** + **j** – **k**

A unit vector orthogonal to the plane is

$$\pm\frac{\mathbf{PQ}\times\mathbf{PR}}{|\mathbf{PQ}\times\mathbf{PR}|} = \pm\frac{1}{\sqrt{18}}(4\mathbf{i}+\mathbf{j}-\mathbf{k})$$

9. A symmetric matrix is of the form

$\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$ where $a, b, c, d, e, f \in$ A.

Thus, the number of symmetric matrices is $3^6 = 729$.

10. A skew symmetric matrix is of the form

$\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$ where $a, b, c \in$ A.

Thus, the number of skew symmetric matrixes is $3^3 = 27$.

11. Put $w = y-x \Rightarrow \frac{dw}{dx} = \frac{dy}{dx} - 1$. Substituting in the given equation, we obtain

$$\frac{dw}{dx} + 1 = x^3w^2 + x^{-1}(w + x)$$

$$\Rightarrow \frac{dw}{dx} = x^3 w^2 + x^{-1}w$$

$$\Rightarrow \frac{1}{w^2}\frac{dw}{dx} - \frac{1}{x}\frac{1}{w} = x^3$$

Put $-\frac{1}{w} = u$, we get

$\frac{du}{dx} + \frac{u}{x} = x^3$ which is a linear equation whose I.F. is x. Multiplying with I.F., we have

$$\frac{d}{dx}(ux) = x^4$$

$$\Rightarrow ux = \frac{x^5}{5} + C$$

$$\Rightarrow -\frac{x}{y-x} = \frac{x^5}{5} + C$$

12. If $y(1) = 2$ then $-\frac{1}{2-1} = \frac{1}{5} + C$

$\Rightarrow C = -\frac{6}{5}$

Thus $-\frac{x}{y-x} = \frac{1}{5}(x^5 - 6)$

$\Rightarrow x = \frac{1}{5}(x^5 - 6)(x - y)$

13. Let the coordinates of R be (h, k), then L is the chord of contact of R with respect to the hyberbola $x^2 - y^2 = 9$.

$\Rightarrow hx - ky = 9$ is same as $x = 9$

$\Rightarrow h = 1, k = 0$ and the coordinates of R are (1, 0).

Now chord of contact of R (1, 0) with respect to the parabola $y^2 = 4(x - 5)$ is

$y \times 0 = 2(x + 1) - 20 \Rightarrow x = 9$.

14. Now $x = 9$ meets the parabola $y^2 = 4(x-5)$ at $T(9, 4)$ and $T'(9, -4)$ and the focus S is on the axis of the parabola at a distance 1 from its vertex (5, 0). So coordinates of S are (6, 0) Area of the triangle $= 4 \times 3 = 12$ sq. units.

15. (a) $\sin 2A + \sin 2B + \sin 2C$ is maximum when the triangle is equilateral. So $A = B = C = 60°$ and $\sin 60° = \sin 120° \Rightarrow \sin A = \sin 2A$

So (a) is true

(b) $\frac{r}{R} = 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \le 4\times\frac{1}{8}$

$\Rightarrow 2r \le R$ or $R \ge 2r$

(c) $\frac{abc}{a+b+c} = \frac{4R\Delta}{2s} = 2R\cdot r \le R^2$

(d) If the triangle is right angled and $A = 90°$. then

$r = (s - a)\tan\frac{A}{2} = s - a = s - 2R$.

$\Rightarrow r + 2R = s$.

16. (a) Shortest distance between L_1 and L_2 is

$$\frac{[\mathbf{j}+\mathbf{k}-(\mathbf{i}+\mathbf{j})]\bullet[(\mathbf{i}-\mathbf{k})\times(\mathbf{i}+\mathbf{j})]}{|(\mathbf{i}-\mathbf{k})\times(\mathbf{i}+\mathbf{j})|}$$

$$\Rightarrow \frac{(\mathbf{k}-\mathbf{i})\bullet(\mathbf{k}-\mathbf{j}+\mathbf{i})}{\sqrt{3}} = 0$$

So the given lines L_1 and L_2 intersect for the point of intersection

$\lambda_1 + 1 = \lambda_2$, $\lambda_2 + 1 = 1$, $\lambda_1 = -1 \Rightarrow \lambda_2 = 0$

and the position vector of the point of intersection is $\mathbf{j} + \mathbf{k}$. Since $(AB) = 0$, position vector of A is $\mathbf{j} + \mathbf{k} \Rightarrow \mathbf{OA} = \mathbf{j} + \mathbf{k}$ and its projection on $\mathbf{r} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ is $\frac{(\mathbf{i}+\mathbf{j})\cdot(\mathbf{i}+\mathbf{j}+\mathbf{k})}{\sqrt{3}} = \frac{2}{\sqrt{3}}$

17. $f'(x) = e^{-x^2}(1-2x^2)$. We know that $e^{-x^2} > 0$ for all $x \in [-1, 1]$. Hence $f'(x) > 0$ if $1-2x^2 > 0$ or $x^2 - \left(\frac{1}{\sqrt{2}}\right)^2 < 0$ i.e. $-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$. Since $-1 < -\frac{1}{\sqrt{2}}$ we get. f strictly increases on $\left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$ and strictly decreases on $\left[-1, -\frac{1}{\sqrt{2}}\right] \cup \left[\frac{1}{\sqrt{2}}, 1\right]$

18. $I = \int \frac{\sin x - \cos x}{\sqrt{\sin x \cos x}}\,dx$

Put $= \cos x + \sin x = t$, so that

$(\cos x - \sin x)\,dx = dt$

$I = -\int \frac{\sqrt{2}dt}{\sqrt{t^2-1}} = -\sqrt{2}\log\left|t + \sqrt{t^2-1}\right| + C$

$= -\sqrt{2}\log\left|\sin x + \cos x + \sqrt{\sin 2x}\right| + C$

19. $|A| = ad - bc$

$\le |ad - bc|$

$\le |ad| + |bc|$

$= |a||d| + |b||c|$

$\le (|a| + |b|)(|c| + |d|)$

Also, $|A| \le |a|\,|d| + |b|\,|c|$

$\le (k)(k) + (k)(k) = 2k^2$

and $|A|^2 \le |a|^2\,|d|^2 + |b|^2\,|c|^2 + 2|a|\,|d|\,|b||c|$

Next, note that

$\left(\sqrt{a^2+b^2}\sqrt{c^2+d^2}\right)^2$

$= (a^2 + b^2)(c^2 + d^2)$

$= a^2d^2 + b^2c^2 + (a^2c^2 + b^2d^2)$

$\ge a^2d^2 + b^2c^2 + 2|a||c||b||d|$

$\ge |A|^2$

$\Rightarrow |A| \le \sqrt{a^2+b^2}\sqrt{c^2+d^2}$

Finally, $\begin{vmatrix} 1 & 1 \\ -2 & 5 \end{vmatrix} = 7$

and max $\{|1|, |1|\}$. max $\{|-2|, |5|\} = (1)(5) = 5$

$\therefore$ (b) is not true.

20. We know that if $a, b > 0$, $a \ne b$, then $A>G>H$ where

$A = \frac{1}{2}(a + b)$, $G = \sqrt{ab}$ and $H = \frac{2ab}{a+b}$.

Also, $AH = ab$.

As A_0, $H_0 < 0$,

$A_1 < H_1 < 0$ and $A_1 H_1 = A_0 H_0 = 51$

$\therefore$ $A_2 < H_2 < 0$ and $A_2 H_2 = A_1 H_1$.

Note that

$2A_1 < A_1 + H_1 \Rightarrow A_1 < \frac{1}{2}(A_1 + H_1) = A_2 < 0$.

and $H_2 = \frac{(-A_1)}{(-A_2)} H_1 > H_1$ $\quad [\because -A_1 > -A_2 > 0]$

Similarly, $A_2 < A_3$, $H_3 > H_2$

Continuing in this way we get

$A_{20} < A_{21} < A_{22}$

and $H_6 > H_5 > H_4$

Also, $A_{99} < A_{100} < A_{101} < H_{101}$

and $A_{100} H_{100} = A_{99} H_{99} = \ldots = A_0 H_0 = 51$