Practice Paper – III

1. Suppose a set A contains 4 elements. The number of relations on A which are not reflexive is
   (a) $(15)(2^{12})$ (b) 2^{12}
   (c) 2^{16} (d) $(35)(2^{6})$

2. The number of integral values of x satisfying the equation
$$\sqrt{x+2-4\sqrt{x-2}}+\sqrt{x+7-6\sqrt{x-2}} = 1$$
   is
   (a) 4 (b) 5
   (c) 6 (d) 7

3. Let $f(x) = \sqrt{1+x\sqrt{1+(x+1)\sqrt{1+(x+2)(x+4)}}}$
   and $\Delta = \begin{vmatrix} f(100) & f(101) & f(102) \\ f(101) & f(102) & f(103) \\ f(102) & f(103) & f(104) \end{vmatrix}$
   then Δ is equal to
   (a) – 104 (b) 0
   (c) 104 (d) 102

4. On the sides AB, BC and CA of a triangle ABC, take $n + 1$, $n + 2$ and $n + 3$ points ($n \in \mathbf{N}$, $n \geq 2$) other than A, B and C. If the number of triangles formed by these $3n + 6$ points is 421, then n is equal to
   (a) 3 (b) 4
   (c) 5 (d) 7

5. Units digit of $\left[\left(10+\sqrt{95}\right)^{24}+\left(10+\sqrt{95}\right)^{76}\right]$, (where $[x]$ denotes the greatest integer $\leq x$) is
   (a) 0 (b) 1
   (c) 8 (d) 9

6. Let z_1 and z_2 be two complex numbers such that $z_1^2 - 4z_2 = 8 - 12i$.
   Also, suppose that x_1 and x_2 are roots of $x^2 - z_1 x + z_2 + ik = 0$,
   where k is a complex number.
   If $|x_1 - x_2| = 6$, then
   (a) k lies on a circle of radius 3.
   (b) k lies on a square of side 3.
   (c) k lies on a circle of radius 6
   (d) k lies on a square of side 6.

7. Negation of $(p \vee \sim q) \wedge r$ is
   (a) $(\sim p \wedge q) \vee r$ (b) $(\sim p \wedge q) \vee \sim r$
   (c) $(\sim p \wedge \sim q) \wedge r$ (d) $(\sim p \wedge \sim q) \vee r$

8. Let x be the arithmetic mean and y, z be the two geometric means between any two positive numbers. Then value of $\dfrac{y^3+z^3}{xyz}$ is
   (a) 2 (b) 3
   (c) $\dfrac{1}{2}$ (d) $\dfrac{3}{2}$

9. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $a, b, c, d \in \mathbf{R}$.
   and $B = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$, $a_1, b_1, c_1, d_1 \in \mathbf{R}$
   Statement-1: If $b_1 = c_1 = 0$, then $AB = BA$
   Statement-2: If $AB = BA$
   $\forall\ a, b, c, d \in \mathbf{R}$, then $b_1 = c_1 = 0$,
   $a_1 = d_1$.

10. **Statement-1:** For every natural number $n \geq 2$.
$$\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}}>\sqrt{n}$$
   Statement-2: For every natural number $n \geq 2$, $\sqrt{n(n+1)} < n + 1$

11. **Statement-1:** If two distinct chords of the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2} = 2$ passing through $(a, -b)$ are bisected by the line $x + y = b$, then $a^2 + 6ab - 7b^2 \geq 0$.

Statement-2: If the tangent and normal to a rectangular hyperbola cut off intercepts a_1 and a_2 on one axis and, b_1 and b_2 on the other, then $a_1 a_2 + b_1 b_2 = 0$

12. **Statement-1:** If the point of intersection of the lines $\frac{x-4}{15} = \frac{y-17}{9} = \frac{z-11}{8}$ and $\frac{x-15}{4} = \frac{y-9}{7} = \frac{z-8}{11}$ lies on the plane $x + y - z = p$, then the value of p is $2b$

Statement-2: The point of intersection of the lines $\frac{x-4}{15} = \frac{y-17}{9} = \frac{z-11}{8}$ and $\frac{x-15}{4} = \frac{y-9}{17} = \frac{z-8}{11}$ is equidistant from the points (4, 17, 11) and (15, 9, 8).

13. A ray of light emerges from the point $P(3, 4)$ and gets reflected upon reaching a plane mirror along the line $x + y - 1 = 0$, at Q (1, 0) Reflection of the point (3, 4) in the reflected ray is the point whose coordinates are

(a) $\left(\frac{4}{5}, \frac{27}{5}\right)$ (b) $\left(\frac{27}{5}, -\frac{4}{5}\right)$

(c) $\left(-\frac{27}{5}, \frac{4}{5}\right)$ (d) None of these

14. Equation of a circle having two of its diameters along the lines $x + 2y = 0$ and $x + 3 = 0$, having radius just sufficient so that it contains the circle $x^2 + y^2 - 4x - 3y = 0$ is

(a) $x^2 + y^2 + 6x - 3y - 54 = 0$
(b) $x^2 + y^2 - 6x + 3y - 54 = 0$
(c) $x^2 + y^2 + 6x - 3y - 45 = 0$
(d) $x^2 + y^2 - 6x + 3y - 45 = 0$

15. If l_1 and l_2 are the length of segments of a focal chord of a parabola, then the length of its latus rectum is

(a) $\frac{l_1 l_2}{l_1 + l_2}$ (b) $\frac{4l_1 l_2}{l_1 + l_2}$

(c) $\frac{1}{l_1} + \frac{1}{l_2}$ (d) $\frac{4l_1 l_2}{|l_1 - l_2|}$

16. Equation of the normal to the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$ perpendicular to the line joining the end points of the semi-major and semi-minor axes of the ellipse in the first quadrant is

(a) $\sqrt{2}\,(5x - 3y) = 16$ (b) $\sqrt{2}\,(5x + 3y) = 16$

(c) $\sqrt{2}\,(3x - 5y) = 9$ (d) $\sqrt{2}\,(3x + 5y) = 9$

17. A line passing through the origin makes angles 45° and 120° with the positive direction of x-axis and y-axis respectively and an acute angle with the positive direction of z-axis, meets the plane $\sqrt{2}\,x - 2y + 2z = 6$ at P. The coordinates of P are.

(a) $(1, -\sqrt{2}, 1)$ (b) $(\sqrt{2}, -1, 1)$

(c) $(1, -1, \sqrt{2})$ (d) $(1, -\sqrt{2}, \sqrt{2})$

18. If $x = a \cos\theta \sin^3 \theta$ and $y = a \sin \theta \cos^3\theta$ then $a^2 (x + y)^2 - 2(x + y)^4$ is equal to

(a) $x^2 - y^2$ (b) $a^2 (x^2 - y^2)$
(c) $x^2 + y^2$ (d) $a^2(x^2 + y^2)$

19. If $\sin^{-1} (x/2) + \sin^{-1} (y/3) = \theta$, then $9x^2 + 12xy \cos\theta + 4y^2$ is equal to

(a) 1 (b) 36
(c) $36 \cos^2\theta$ (d) $36 \sin^2\theta$

20. If the angles of elevation of the top of a tower from the foot and top of a pole of height a standing on the same ground as the tower are respectively 60° and 30°, then the height of the tower is

(a) $2a$ (b) $3a/2$
(c) $\sqrt{3}\,a$ (d) $3a$

21. The value of $\int_{-\pi}^{\pi} \frac{\cos^4 x}{1 + a^x} dx$ $(a > 0)$ is

(a) $\frac{3\pi}{8}$ (b) $\frac{\pi}{8}$

(c) $\frac{5\pi}{8}$ (d) $\frac{\pi}{1 + a^\pi}$

22. Knowing that $\lim_{x\to 0+} x^x = 1$,

$$\lim_{x\to 0+}\left(\lim_{x\to\infty} \frac{[1^2 x^x] + [2^2 x^x] + \cdots + [n^2 x^x]}{n^3}\right)$$

is equal to

(a) 2/3 (b) 1/3
(c) 1 (d) 5/3

23. Let $f(x) = \begin{cases} |x-2|[x] & , \quad x \ge 1 \\ \sin\left(\frac{\pi}{2}x\right) & , \quad x < 1 \end{cases}$

where [,] is the greatest integer, function. Then f is

(a) f is not continuous at $x = 1$
(b) f is differentiable at $x = 1$
(c) f is not differentiable at $x = 3/2$
(d) f is continuous but not differentiable at $x = 1$

24. The triangle formed by the tangent to the curve $y = x^2 + b(x - 1)$ at the point (1, 1) and the coordinate axes, lies in the first quadrant. If its area is 2, then the value of b is

(a) – 1 (b) 3
(c) – 3 (d) 1

25. $f(x) = |x \log x|$ $(x > 0)$ is monotonically decreasing in

(a) $\left(0, \frac{1}{10}\right)$ (b) $\left[\frac{1}{e}, 1\right]$

(c) (1, 10) (d) (e, ∞)

26. The value of $\int_{-1}^{1} \max\{2-x, 2, 1+x\}\, dx$ is
    (a) 1/2 (b) 5/2
    (c) 7/2 (d) 9/2
27. The solution of the differential equation $(xy^4 + 2y)dx - 2xdy = 0$ is
    (a) $4x^4y^3 + 3x^3 = cy^3$ (b) $3x^3y^4 - 8x^3 = cy^3$
    (c) $3x^4y^3 + 8x^3 = cy^3$ (d) $4x^3y^4 + 4y^3 = cx^3$
28. Let **a**, **b**, **c** be three non-zero vectors such that no two of them are collinear and $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} + \frac{1}{\sqrt{2}} |\mathbf{a}|\,|\mathbf{c}|\,\mathbf{b} = 0$.
    Then the angle between **a** and **c** is
    (a) $3\pi/4$ (b) $\pi/4$
    (c) $\pi/2$ (d) π
29. Consider the function $f(x) = \sin kx + \{x\}$, where $\{x\}$ is the fractional part of x

    Statement-1: f is periodic for all k of the form π/n, $n \in \mathbf{N}$

    Statement-2: sum of two periodic function f_1 and f_2 is periodic with period *l.c.m* of periods if f_1 and f_2
30. If the mean of a data is 4 and $A = 3\Sigma(x_i - 3)^2$, $B = 2\Sigma(x_i - 4)^2$, $C = 4\Sigma(x_i - 5)^2$ then

    Statement-1: Min $(A, B, C) = B$

    Statement-2: Sum of the squares of the deviations is least when taken from mean

Answers

1. (a)	**2.** (c)	**3.** (b)	**4.** (a)
5. (d)	**6.** (a)	**7.** (b)	**8.** (a)
9. (d)	**10.** (b)	**11.** (b)	**12.** (a)
13. (b)	**14.** (c)	**15.** (b)	**16.** (a)
17. (b)	**18.** (d)	**19.** (d)	**20.** (b)
21. (a)	**22.** (b)	**23.** (d)	**24.** (c)
25. (b)	**26.** (d)	**27.** (c)	**28.** (a)
29. (c)	**30.** (a)		

Hints and Solutions

1. $A \times A$ contain $4 \times 4 = 16$ elements.

   Number of relations that can be defned on A is 2^{16}.

   A reflexive relation on A must contain all the four elements of the form (a, a), $a \in A$ and any number of remaining 12 elements from $A \times A$. Thus, number of reflexive relation on A is 2^{12}.

   Hence, number of relations on A which are not reflexive is $2^{16} - 2^{12} = (15)(2^{12})$

2. Note that $x \ge 2$. Let $t = \sqrt{x-2}$, the given equation becomes

   $\sqrt{t^2 + 4 - 4t} + \sqrt{t^2 + 9 - 6t} = 1$

   $\Rightarrow |t - 2| + |t - 3| = 1$

   $\Rightarrow 2 \le t \le 3 \Rightarrow 2 \le \sqrt{x-2} \le 3$

   $\Rightarrow 6 \le x \le 11$

   Thus, number of integral values of x satisfying the given equation is 6.

3. $1 + (x + 2)(x + 4) = 1 + x^2 + 6x + 8 = (x + 3)^2$

   $\therefore\ 1 + (x + 1)\sqrt{1 + (x+2)(x+4)}$

   $= 1 + (x + 1)(x + 3) = x^2 + 4x + 4 = (x + 2)^2$

   $\Rightarrow 1 + x\sqrt{1 + (x+1)\sqrt{1 + (x+2)(x+4)}}$

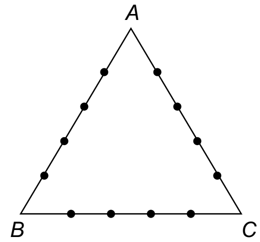
   $= 1 + x(x + 2) = (x + 1)^2$

   Thus, $f(x) = x + 1$, and

   $$\Delta = \begin{vmatrix} 101 & 102 & 103 \\ 102 & 103 & 104 \\ 103 & 104 & 105 \end{vmatrix} = 0$$

   [use $R_3 \to R_3 - R_2$ and $R_2 \to R_2 - R_1$]

4. Number of triangles that can be formed with $3n + 6$ points is



   $N = (n + 1)(n + 2)(n + 3) + ({}^{n+1}C_2)(n + 2 + n + 3)$
   $+ ({}^{n+2}C_2)(n + 1 + n + 3) + ({}^{n+3}C_2)(n + 1 + n + 2)$
   $= (n + 1)(n + 2)(n + 3) + \frac{1}{2}(n + 1)(n)(2n + 5)$
   $+ \frac{1}{2}(n + 2)(n + 1)(2)(n + 2) + \frac{1}{2}(n + 3)(n + 2)$
   $(2n + 3)$

   Put $n + 2 = m$, so that

   $N = (m - 1)(m)(m + 1) + \frac{1}{2}(m - 1)(m - 2)(2m + 1)$
   $+ m^2(m - 1) + \frac{1}{2}(m + 1)m(2m - 1)$

$= m^3 - m + m^3 - m^2 + 2m^3 - 2m^2 + 1$

$= 4m^3 - 3m^2 - m + 1$

Now, $4m^3 - 3m^2 - m + 1 = 421$

$\Rightarrow 4m^3 - 3m^2 - m - 420 = 0.$ (1)

Clearly, $4m^3 > 420 \Rightarrow m^3 > 105.$

$\Rightarrow m \geq 5.$

Also, (1) is satisfied by $m = 5$.

$\therefore\ n + 2 = 5 \Rightarrow n = 3.$

5. For $k \in \mathbf{N}$, k even,

$N(k) = (10 + \sqrt{95})^k + (10 - \sqrt{95})^k$

$= 2[{}^kC_1\ 10^k + {}^kC_2\ (10)^{k-2}\ (95) + \ldots + {}^kC_k\ (95)^{k/2}]$

$\Rightarrow N(k)$ is divisible by 10.

Let $N(24) = 10m_1$

and $N(76) = 10m_2$

$\Rightarrow (10 + \sqrt{95})^{24} + (10 + \sqrt{95})^{76} + M$

$= 10\ (m_1 + m_2)$

where $M = (10 - \sqrt{95})^{24} + (10 - \sqrt{95})^{76}$

But $10 - \sqrt{95} = \dfrac{5}{10+\sqrt{95}} < \dfrac{1}{3}$

$\Rightarrow 0 < M < 1.$

$\Rightarrow$ Units digit of $[(10 + \sqrt{95})^{24} + (10 + \sqrt{95})^{76}]$ is 9

6. $x_1 + x_2 = z_1$, $x_1\ x_2 = z_2 + ik$

Also,

$(x_1 - x_2)^2 = (x_1 + x_2)^2 - 4x_1\ x_2$

$= z_1^2 - 4z_2 - 4ik$

$= 8 - 12i - 4ik$

$|x_1 - x_2|^2 = |8 - 12i - 4ik|$

$\Rightarrow 36 = |-4i|\ |k + 3 - 2i|$

$\Rightarrow |k - (-3 + 2i)| = 9$

Thus, k lies on a circle of radius 3.

7. $\sim [(p \vee \sim q) \wedge \text{r}]$

$\equiv \sim (p \vee \sim q) \vee (\sim r)$

$\equiv (\sim p \wedge q) \vee \sim r$

8. Let two positive numbers be a and b. Then $x = (a + b)/2$. Also, a, y, z, b are in G.P. If r is the common ratio of this G.P., then $b = ar^3 \Rightarrow r = (b/a)^{1/3}$. we have

$$\frac{y^3 + z^3}{xyz} = \frac{a^3r^3 + a^3r^6}{x(ar)(ar^2)} = \frac{a(1+r^3)}{x} = \frac{a+b}{(a+b)/2} = 2$$

9. If $AB = BA\ \forall\ A$, then

$$\begin{bmatrix}1 & 0\\1 & 0\end{bmatrix}B = B\begin{bmatrix}1 & 0\\1 & 0\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}a_1 & b_1\\a_1 & b_1\end{bmatrix} = \begin{bmatrix}a_1+b_1 & 0\\c_1+d_1 & 0\end{bmatrix}$$

$\Rightarrow b_1 = 0,\ a_1 = c_1 + d_1$

Next

$$\begin{bmatrix}0 & 1\\0 & 1\end{bmatrix}B = B\begin{bmatrix}0 & 1\\0 & 1\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}c_1 & d_1\\c_1 & d_1\end{bmatrix} = \begin{bmatrix}0 & a_1+b_1\\0 & c_1+d_1\end{bmatrix}$$

$\Rightarrow c_1 = 0,\ a_1 + b_1 = d_1$

Thus, $a_1 = d_1$, $b_1 = c_1 = 0$.

$\therefore$ Statement-2 is true.

When $b_1 = c_1 = 0$, but $a_1 \neq d_1$

$AB \neq BA\ \forall\ A$

Thus, statement-1 is false.

10. As $n < n + 1\ \forall\ n \geq 2$

$\Rightarrow n(n + 1) < (n + 1)^2\ \forall\ n \geq 2$

$\Rightarrow \sqrt{n(n+1)} < n + 1\ \forall\ n \geq 2$

Thus, statement-2 is true.

Also, for $n \geq 2$

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}}$$

$$> \underbrace{\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + \cdots + \frac{1}{\sqrt{n}}}_{n \text{ times}}$$

$$\Rightarrow \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} > \sqrt{n}$$

$\therefore$ Statsement-1 is true.

Thus, both the statements are correct but statement-2 is not correct explanation of statement-1.

11. In statement-1, let $(t, b - t)$ be any point on the line $x + y = b$.

Equation of the chord of the ellipse whose mid point is $(t,\ b - t)$, is $\dfrac{tx}{2a^2} + \dfrac{y(b-t)}{2b^2} - 1 = \dfrac{t^2}{2a^2} + \dfrac{(b-t)^2}{2b^2} - 1$ (1)

If $(a, -b)$ lies on (1), then

$$\frac{ta}{2a^2} - \frac{b(b-t)}{2b^2} = \frac{t^2}{2a^2} + \frac{(b-t)^2}{2b^2}$$

$\Rightarrow t^2 (a^2 + b^2) - ab(3a + b)t + 2a^2 b^2 = 0$

Since t is real,

$a^2 b^2 (3a + b)^2 - 4 (a^2 + b^2)(2a^2 b^2) \geq 0$

$\Rightarrow a^2 + 6ab - 7b^2 \geq 0$

So, statement–1 is true.

In statement-2, let the equation of the rectangular hyperbola be $x^2 - y^2 = a^2$ and $P(a \sec \theta, a \tan \theta)$ be any point on it.

Equations of the tangent and normal at P to the hyperbola are respectively

$x \sec \theta - y \tan \theta = a$ and $x \cos \theta + y \cot \theta = 2a$

If a_1, a_2 are the intercepts on x-axis, then

$a_1 = a \cos \theta$, $a_2 = 2a/\cos \theta \Rightarrow a_1 a_2 = 2a^2$

similarly $b_1 b_2 = -2a^2$, Hence $a_1 a_2 + b_1 b_2 = 0$

So that statement-2 is also true but does not lead to statement-1.

12. Any point on the line $\dfrac{x-4}{15} = \dfrac{y-17}{9} = \dfrac{z-11}{8}$

is $(15r + 4, 9r + 17, 8r + 11)$ and on the line.

$\dfrac{x-15}{4} = \dfrac{y-9}{17} = \dfrac{z-8}{11}$ is $(4r' + 15, 17r' + 9, 11r' + 8)$

where r and r' are the distances of the points from the points (4, 17, 11) and (15, 9, 8) respectively.

If these two points are same the

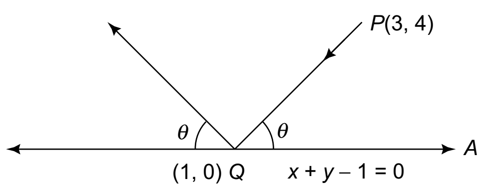
$15r + 4 = 4r' + 15$, $9r + 17 = 17r' + 9$, $8r + 11 = 11r' + 8$

$\Rightarrow r = r' = 1$

which shows that statement-2 is true using it in statement-1, the point of intersection is (19, 26, 19). If it lies on $x + y - z = p$

Then $p = 26$ and the statement-1 in also true.

13. The ray through $P(3, 4)$ meets the line $x + y - 1 = 0$ at Q (1, 0)



If PQ makes an angle θ with the line $x + y - 1 = 0$, the reflected ray makes an angle $-\theta$, with the same line

$\tan \theta = \dfrac{2+1}{1+2(-1)} = -3$

Let the slope of the reflected ray be m

then $\tan(-\theta) = \dfrac{m+1}{1-m} \Rightarrow m = \dfrac{1}{2}$

and the equation of the reflected ray is $x - 2y - 1 = 0$ let (α, β) be the image of (3, 4) in $x - 2y - 1 = 0$

then $\dfrac{\alpha+3}{2} - 2 \times \dfrac{\beta+4}{2} - 1 = 0 \Rightarrow \alpha - 2\beta - 7 = 0$

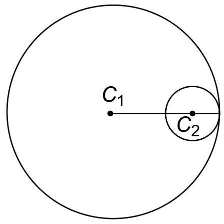
and $\dfrac{\beta-4}{\alpha-3} \times \dfrac{1}{2} = -1 \Rightarrow 2\alpha - \beta - 10 = 0$

$\Rightarrow \alpha = \dfrac{27}{5}, \beta = \dfrac{-4}{5}$

14. Centre of the required circle is $C_1 (-3, 3/2)$. Centre of the given circle is $C_2 (2, 3/2)$ and radius is $r = 5/2$

Radius of the required circle is $|C_1 C_2| + r = 5 + \dfrac{5}{2}$

$= \dfrac{15}{2}$

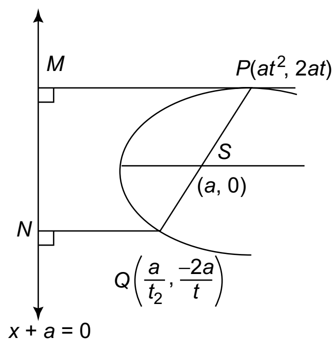


and hence its equation is $x^2 + y^2 + 6x - 3y - 45 = 0$

15. Let PSQ be the focal chord of the parabola $y^2 = 4ax$.

$P(at^2, 2at)$, $Q(a/t^2, -2a/t)$ $S(a, 0)$

$l_1 = PS = PM = at^2 + a$



$l_2 = QS = QN = a/t^2 + a.$

$\dfrac{l_1 l_2}{l_1 + l_2} = a$

16. Slope of the line joining $A(5, 0)$ and $B(0, 3)$ is $-3/5$ slope of the normal is $5/3$.

Equation of the normal at $P(5\cos\theta, 3\sin\theta)$ on the ellipse is

$$\frac{5x}{\cos\theta} - \frac{3y}{\sin\theta} = 25 - 9 = 16$$

$$\Rightarrow \frac{5\sin\theta}{3\cos\theta} = \frac{5}{3} \Rightarrow \tan\theta = 1 \Rightarrow \theta = \pi/4$$

so the required equation is

$$\sqrt{2}\,(5x - 3y) = 16$$

17. If it makes an angle θ with the positive direction of z-axis then $\cos^2 45° + \cos^2 120° + \cos^2\theta = 1 \Rightarrow \theta = 60°$

Equation of the line is

$$\frac{x}{\cos 45°} = \frac{y}{\cos 120°} = \frac{z}{\cos 60°} \text{ or } \sqrt{2}\,x = -2y = 2z$$

Any point on this line is $\left(r/\sqrt{2}, -r/2, r/2\right)$ which lies on $\sqrt{2}\,x - 2y + 2z = 6$

$\Rightarrow r = 2$ and the required coordinates are $P(\sqrt{2}, -1, 1)$

18. $x + y = a\cos\theta\sin\theta$, $x^2 + y^2 = a^2\cos^2\theta\sin^2\theta$

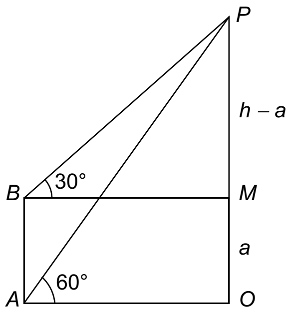
$$(1 - 2\sin^2\theta\cos^2\theta) = (x + y)^2\left(1 - 2\frac{(x+y)^2}{a^2}\right)$$

19. $\cos\theta = \sqrt{1 - (x^2/4)}\sqrt{1 - (y^2/9)} - xy/6$

$\Rightarrow (4 - x^2)(9 - y^2) = (xy + 6\cos\theta)^2$

$\Rightarrow 9x^2 + 12xy\cos\theta + 4y^2 = 36\sin^2\theta.$

20. $(h - a)\cot 30° = BM = AO = h\cot 60°$



$\Rightarrow h(\cot 30° - \cot 60°) = a(\cot 30°)$

$$\Rightarrow h\left(\sqrt{3} - \frac{1}{\sqrt{3}}\right) = a\sqrt{3}$$

$\Rightarrow h = 3a/2$

21. $$I = \int_{-\pi}^{\pi}\frac{\cos^4 x}{1 + a^x}dx = \int_{-\pi}^{\pi}\frac{\cos^4 x}{1 + a^{-x}}dx = \int_{-\pi}^{\pi}\frac{a^x}{1 + a^x}\cos^4 x\,dx$$

$$2I = \int_{-\pi}^{\pi}\left(\frac{\cos^4 x}{1 + a^x} + \frac{a^x}{1 + a^x}\cos^4 x\right)dx$$

$$= \int_{-\pi}^{\pi}\cos^4 x\,dx = 2\int_0^{\pi}\cos^4 dx = 4\int_0^{\pi/2}\cos^4 x\,dx$$

$$= 4\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}$$

$$I = \frac{3}{8}\pi.$$

22. $1^2 x^x - 1 < [1^2 x^x] \le 1^2 x^x$; $2^2 x^x - 1 < [2^2 x^x] \le 2^2 x^x$; ... ; $n^2 x^x - 1 < [n^2 x^x] \le n^2 x^2$. Adding, we get

$$\frac{x^x \sum_{j=1}^{n} j^2 - n}{n^3} < \frac{\sum_{j=1}^{n}[j^2 x^x]}{n^3} \le \frac{x^x \sum_{j=1}^{n} j^2}{n^3}$$

$$\Rightarrow x^x\frac{n(n+1)(2n+1)}{6n^3} - \frac{1}{n^2} < \frac{\sum_{j=1}^{n}[j^2 x^x]}{n^3} \le x^x\frac{n(n+1)(2n+1)}{6}$$

Taking limit $n \to \infty$

$$x^x\cdot\frac{1}{3} \le \lim_{n\to\infty}\frac{\sum_{j=1}^{n}[j^2 x^x]}{n^3} \le x^x\cdot\frac{1}{3}$$

$$\text{Thus } \lim_{n\to\infty}\frac{\sum_{j=1}^{n}[j^2 x^x]}{n^3} = x^x\cdot\frac{1}{3}$$

$$\lim_{x\to 0+}\lim_{n\to\infty}\frac{\sum_{j=1}^{n}[j^2 x^x]}{n^3} = \frac{1}{3}$$

23. $$\lim_{x\to 1-} f(x) = \lim_{x\to 1-}\sin\left(\frac{\pi}{2}x\right) = 1$$

$$\lim_{x\to 1+} f(x) = \lim_{x\to 1+}|x - 2|[x] = 1$$

$$f'(1+) = \lim_{h\to 0+}\frac{f(1+h) - f(1)}{h} = \lim_{h\to 0+}\frac{|1 + h - 2|[1 + h] - 1}{h}$$

$$= \lim_{h\to 0+}\frac{|-1 + h|\,|1 + h| - 1}{h}$$

$$= \lim_{h\to 0+}\frac{1 - h - 1}{h} = -1$$

$$f'(1-) = \lim_{h\to 0-}\frac{f(1+h) - f(1)}{h} = \lim_{h\to 0-}\frac{\sin\frac{\pi}{2}(1 + h) - 1}{h}$$

$$= \lim_{h\to 0-}\frac{\cos\frac{\pi}{2}h - 1}{h}$$

$$= -\lim_{h\to 0-}\frac{2\sin^2\frac{\pi}{4}h}{h} = 0.$$

f is not differentiable at $x = 1$

$f'(3/2+) = -1 = f'(3/2-)$

24. Equation of tangent at (1, 1) is

$Y - 1 = (2 + b)(X - 1)$

This meets X-axis at $\left(-\dfrac{1}{2+b}+1, 0\right)$ and y-axis at $(0, 1 - (2 + b))$

So area of the triangle = $\dfrac{1}{2}\left(\dfrac{1+b}{2+b}\right)(-(1+b))$

Thus $-(1 + b)^2 = 4(2 + b) \Rightarrow (b + 3)^2 = 0$
$\Rightarrow b = -3.$

25. $f(x) = \begin{cases} x\log_{10} x, x > 1 \\ -x\log_{10} x., x \le 1 \end{cases}$

$$f'(x) = \begin{cases} -\dfrac{1}{\log 10}[1+\log x], & x > 1 \\ -\dfrac{1}{\log 10}[1+\log x], & x < 1 \end{cases}$$

Hence for $x < 1$ $f'(x) < 0$ if $1 + \log x > 0$

$\Rightarrow \log x > -1 \Rightarrow x > 1/e$

26. $f(x) = \begin{cases} 2-x &, -1 \le x \le 0 \\ 2 &, 0 \le x \le 1 \end{cases}$

$$\int_{-1}^{1} f(x)\,dx = \int_{-1}^{0}(2-x)\,dx + \int_{0}^{1} 2\,dx = 2 + \frac{1}{2} + 2 = \frac{9}{2}.$$

27. $xy^4\,dx + 2y\,dx - 2xdy = 0$

$$xdx + \frac{2}{y^2}\,\frac{y\,dx - xdy}{y^2} = 0$$

$$x^3\,dx + 2\left(\frac{x}{y}\right)^2 d\left(\frac{x}{y}\right) = 0$$

$$\Rightarrow \frac{x^4}{4} + \frac{2}{3}\left(\frac{x}{y}\right)^3 = c$$

$\Rightarrow 3x^4y^3 + 8x^3 = cy^3.$

28. $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{a}.\mathbf{c})\mathbf{b} - (\mathbf{b}.\mathbf{c})\mathbf{a} = -\dfrac{1}{\sqrt{2}}\,|\mathbf{a}|\,|\mathbf{c}|\mathbf{b}$

$\Rightarrow \left(\mathbf{a}\,.\,\mathbf{c} + \dfrac{1}{\sqrt{2}}|\mathbf{a}|\,|\mathbf{c}|\right)\mathbf{b} = (\mathbf{b}.\mathbf{c})\mathbf{a}$

$\Rightarrow (\mathbf{a}.\mathbf{c}) = -\dfrac{1}{\sqrt{2}}\,|\mathbf{a}|\,|\mathbf{c}| \Rightarrow \cos\theta = -\dfrac{1}{\sqrt{2}},$

where θ is the angle between **a** and **c**. So

$\theta = \dfrac{3\pi}{4}.$

29. Period of $\{x\}$ is 1 and period of $\sin\dfrac{\pi}{n}x$ is $\dfrac{2\pi n}{\pi} = 2n$. Hence period of $f(x)$ is $2n$.

In general, period of sum two functions may not be defined e.g. $f_1(x) = \sin x$, $f_2(x) = \{x\}$ *l.c.m* of $(2\pi, 1)$ is not defined

30. Clearly statement-2 implies statement-1